EUCLID

Taken from a Braſs Coin in the Repoſitory of the late Queen Chriſtian of Sweden

Euclid *the Mathematician, was of* Alexandria, *where he Taught in the Reign of* Ptolemy Lagus *in the* CXX Olympiad, *and Year of* Rome 454. *He Wrote many things relating to Muſick and Geometry: But his* XV *Books of Elements (of which he is generaly thought to be only the Collector) are most applauded: the two last are attributed to* Hypſicles *of* Alexandria, *and not to him.* *Cardan Voſsius.*

THE
ELEMENTS
OF
EUCLID;

With Select THEOREMS out of

ARCHIMEDES.

By the Learned ANDREW TACQUET.

To which are added,
Practical COROLLARIES, ſhewing the
Uſes of many of the Propoſitions.

The Whole abridg'd, and publiſh'd in *Engliſh*,
By *WILLIAM WHISTON*, M. A.

The SIXTH EDITION.

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see Whiston's
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AN HISTORICAL ACCOUNT OF THE Riſe and Progreſs OF THE MATHEMATICKS.

IT ſeem'd meet to me, when I was about to ſet forth the Elements of the Mathematicks, to premiſe a few Things concerning the Riſe and Excellency of this Science, that its Candidates may underſtand what a Kind of Science it is to which they are about to dedicate themſelves; and that it may be made manifeſt againſt thoſe who ſlight thoſe Things whereof they are ignorant, of how great Value and Dignity this Knowledge is, which the wiſeſt Men of all Ages have, with incredible Study, labour'd to attain unto, and become poſſeſs'd of. Moreover, I muſt own that *Peter Ramuſ*'s Labours have been of great Service to me in the compiling of this Account, who in the whole firſt Book of his Inſtitution, which is not a little one, hath out of ***Proclus***, ***Laertius***, ***Gellius***, ***Polybius***, ***Tzetzes***, and others, compos'd a Mathematical Hiſtory both accurately and copiouſly.

The Mathematical Sciences were the firſt of all others amongſt Men, if we may believe *Joſephus*. He, *Book* I. *Chap*. 3, writeth, that the Poſterity of *Seth* obſerv'd the Order of the Heavens, and the Courſes of the Stars. And leſt theſe Inventions ſhould ſlip out of the Knowledge of Men, *Adam* having predicted a twofold Deſtruction of the Earth, one by a Deluge, the other by Fire, they rais'd two Columns, one of Brick, of Stone the other; and inſcribed their Inventions upon them, that, if the Brick one ſhould happen to be deſtroy'd by the Deluge, that of Stone which would remain might afford Men an Opportunity of being inſtructed, and preſent to their View the Things which it had inſcribed on it. They ſay alſo that that ſtone Pillar, which even in our Days is ſeen in *Syria*, was dedicated by them. This *Joſephus* ſays: whom I leave to vouch for the Story.

That the *Aſſyrians* and *Chaldeans* were the firſt of Mortals, after the Flood, who applied themſelves to the Mathematicks, is delivered by the ſame *Joſephus*; as alſo by *Pliny*, *Diodorus*, and *Cicero*. But the Mathematic Arts which firſt ſprang amongſt the *Chaldeans*, amongſt whom they flouriſhed, were afterwards transferr'd out of *Chaldea* and *Aſſyria* unto the *Egyptians*, by *Abraham*. For, when, at the Command of God, he went forth from his native Soil into *Paleſtine*, and from thence into *Egypt*, and perceiv'd the *Egyptians* to be taken with the Study of good Arts, and to be of a very notable Wit and Capacity for Learning (as *Joſephus* teſtifies, *Book* I. *Chap*. 9.); he communicated to them Arithmetic and Aſtronomy; and conſequently Geometry, which muſt of Neceſſity go before Aſtronomy. In which Studies afterwards the *Egyptians* ſo flouriſhed, that *Ariſtotle*, 1 *Metaph. Chap*. 1, doth affirm, That the Mathematic Arts were firſt found out in *Egypt*, by their Prieſts; who by their Employment were at Leiſure for theſe Things.

Then theſe Arts, croſſing the Sea out of *Egypt*, came to the Philoſophers of *Greece*: For *Thales* the *Mileſian*, who flouriſh'd 584 Years before Chriſt, was the firſt of the *Greeks*, who, coming into *Egypt*, transferr'd Geometry from thence into *Greece*. He it was indeed, who, beſides other Things, found out the 5th, 15th, and 26th Propoſitions of the firſt Book. To the ſame are alſo owing the 2d, 3d, 4th, and 5th, of the Fourth Book. The ſame Perſon began to obſerve the Equinoxes and Solſtices, as *Laertius* teſtifies; and he was the firſt who foretold an Eclipſe of the Sun, as *Hippias* and *Ariſtotle* do write; and *Tzetzes* ſaith, That he alſo foreſhew'd an Eclipſe of the Moon to King *Cyrus*. For which Things ſake he is to be look'd on as the firſt Founder and Author of the Mathematical Sciences in *Greece*.

After him was *Pythagoras* of *Samos*: Which moſt ancient Philoſopher exceedingly improv'd and adorn'd the Mathematic Sciences. And he ſo gave himſelf to Arithmetic in particular, that almoſt his whole Method of Philoſophizing was taken from Numbers. And he firſt of all, as *Laertius* relates, abſtracted Geometry from Matter; in which Elevation of the Mind, he found out the 32d, 44th, 47th, and 48th Propoſitions of the Firſt Book. But he is eſpecially celebrated for the Invention of *Prop.* 32. and 47. of that Book; and he conceiv'd ſo great Joy upon this Invention, that, as *Apollodorus* witneſſes in *Laertius*, on that Account he ſacrific'd an Hecatomb. The ſame Perſon firſt laid open the Matter of incommenſurable Magnitudes, and the Five regular Bodies. The ſame Perſon did both moſt diligently teach and exerciſe the Art of Aſtrology and Muſic: For he did not only acutely and ſubtly find out many Things himſelf, but he alſo firſt opened a School, in which Youth might learn theſe honourable and noble Arts.

Pythagoras was followed by *Anaxagoras* of *Clazomenæ*, and *Oenopides* of *Chios*, of whom *Plato* makes mention in his Dialogue, *The Lovers*, where Young Men are brought in contending about *Anaxagoras* and *Oenopides* in their Deſcriptions of Circles. *Ariſtotle* reports, that a certain Treatiſe of Geometry was written by *Anaxagoras*; and we have it from *Laertius*, that it was ſhew'd by him that the Sun is greater than *Peloponneſus* (a notable Inſtance of the Infancy of Aſtronomy at that Time); and that he made ſome Conjectures concerning Habitations in the Moon. As for *Oenopides*, to him *Proclus* aſcribes the 12. and 13. *l.* 1.

Theſe were followed by *Briſo*, *Antipho*, and *Hippocrates* of *Chios*, all of them, for attempting the Quadrature of the Circle, reprehended by *Ariſtotle*, and at the ſame time celebrated. But, amongſt them, *Hippocrates* was by far the moſt famous; that celebrated Perſon, who, of a Merchant growing to be a Philoſopher and Geometrician, beſides the Quadrature of the Circle, alſo firſt attempted the Doubling of the Cube, by two mean Proportionals, which as being an excellent, and indeed the only Way, all that have followed him to this Time have embrac'd it. 'Tis alſo his peculiar and great Commendation, that he, as *Proclus* teſtifies, firſt wrote Elements, and digeſted into Order the Diſcoveries made by others.

Democritus was admirable, not in Philoſophy only, but alſo in the Mathematicks. His Phyſical Monuments, and, if ſuch there were, his Mathematical Works alſo, are wholly loſt, through the Envy (as ſome report) of *Ariſtotle*, who deſired to have no other Writings read but his own. The Philoſophy of *Democritus* hath been reſtored by *Peter Gaſſendus*, in a moſt learned Work lately put forth. *Theodorus Cyrenæus*, although none of his Mathematical Inventions are extant, yet is great upon this Account, if there were no other, that he is reported to have been the Maſter of *Plato*.

Unto *Plato* therefore we are come at length, than whom no one brought greater Luſtre to the Mathematical Sciences. He amplified Geometry with great and notable Additions, beſtowing incredible Study upon it. And above all, the Art Analytic, or of Reſolution, was found out by him, the moſt certain Way of Invention and Reaſoning. He ſet off and illuſtrated his Books of Philoſophy in a Mathematical Way, and encourag'd whatſoever was admirable in Mathematical Philoſophy. Upon the Door of his Academy was read this Inſcription: οὐδεὶς ἀγεωμέτρητος εἰσίτω: *Let no one ignorant of Geometry enter here*: An illuſtrious Inſtance to demonſtrate, how the Mathematicks are not foreign but proper, not unuſeful, or unbecoming, but honourable and profitable to ſound and certain Philoſophy. In a word, how great both Admirer and Maſter of the Mathematicks *Plato* was, that Man will of himſelf eaſily underſtand, who ſhall read his Monuments through.

Out of *Plato*'s Academy, almoſt innumerable Mathematicians came forth. Thirteen of *Plato*'s familiar Acquaintance are commemorated by *Proclus*, as Men by whoſe Studies the Mathematicks were improv'd. From hence were *Leodamus* the *Thaſian*, *Archytas* the *Tarentine*, *Theætetus* the *Athenian*, by whom the Mathematicks were notably enlarged. *Leodamus* practiſed the Analyſis received from *Plato*, and is ſaid by *Laertius* to have found out many Things by the Help of it. As for *Theætetus*, both his own Inventions, amongſt which are the Elements written by him, and the Inſcription of regular Bodies; and *Plato*'s Encomiums, who alſo inſcribed a Dialogue to his Name, do make him famous.

Archytas alſo wrote Elements himſelf; and his Doubling of the Cube is mentioned by *Eutocius*; whoſe ſingular Commendation it likewiſe was, that he was almoſt the Firſt that brought down the Mathematicks to human Uſes; by whoſe Contrivance alſo a wooden Pigeon was made to fly, as *Gellius* reports; he being followed by *Dædalus*, and other Artificers, yielded Matter for the Fables of the Poets. Moreover, *Archytas* was both a Mathematician and General of an Army: He five Times commanded the Forces of his own Citizens, in the Wars of his Country, and five Times overcame their Enemies. The meer Name of *Neoclides* is only famous, he being more illuſtrious for his Scholar *Leon* perhaps, than for his own Inventions. *Leon* certainly wrote Elements of all the Mathematicks, improv'd them, and made them more fit for Uſe. Wherefore he is deſervedly to be reckon'd amongſt the chief Compilers of Elements.

Eudoxus of *Cnidos* was not inferior to *Leon:* A Man great in Arithmetic, and to him (if we may believe the *Greek* Scholiaſt) we owe the whole Fifth Book. He likewiſe wrote Elements, and made them more general, and increas'd the Sections begun by *Plato*; over and above this he was the firſt Framer of Aſtronomical Hypotheſes, and derived down the Springs of Geometry, as *Archytas* had done before, to Mechanicks. *Amyclas* the *Heracleot*, and *Menæchmus*, and his Brother *Dinoſtratus*, *Helicon* of *Cyzium*, *Theudius*, *Hermotimus* the *Colophonian*, *Philippus* the *Medmæan*, all *Platoniſts*,

niſts, rendered Geometry much more perfect. *Menæchmus* alſo found out the Conic Sections, and by the Help of them two mean Proportionals; whoſe Invention in this Caſe is preferr'd by *Eutocius* before any other. *Theudius* and *Hermotimus* made the Elements more univerſal and full. And all theſe, who were of *Plato*'s Academy, brought Mathematic Philoſophy to Perfection, as *Proclus* ſaith. *Xenocrates* alſo, one of *Plato*'s Auditors, and Maſter of *Ariſtotle*, as well as *Ariſtotle* himſelf, were famous for the Knowledge of the Mathematicks. When a certain Perſon, who knew nothing of Geometry, was minded to be his Auditor, *Go thy way*, ſaith he, *for thou wanteſt the very Handles of Philoſophy*.

But of *Ariſtotle*, what can I ſay? All his Books are filled with Mathematical Arguments, out of a Collection of which *Blancane* hath made a Book. Two of *Ariſtotle*'s School are eſpecially celebrated, *Eudemus* and *Theophraſtus*. This latter wrote two Books of Numbers, four of Geometry, and one of indiviſible Lines: The other compos'd a Mathematical Hiſtory; and from him *Proclus* and others have borrow'd theirs. To *Ariſteus*, *Iſidore*, *Hypſicles*, moſt ſubtle Geometricians, we are eſpecially indebted for the Books of Solids. Laſtly, *Euclid* gathered together the Inventions of others, diſpos'd them into Order, improv'd them, and demonſtrated them more accurately, and left to us thoſe *Elements*, by which Youth is every where inſtructed in the Mathematicks. He died in the Year before Chriſt 284. There follow'd *Euclid* almoſt an 100 Years afterwards

Eratoſthenes and *Archimedes.* The Name of *Eratoſthenes* was very famous, but his Writings are loſt. Many Remains we have of *Archimedes*, and many we have loſt.

But, when I name *Archimedes*, I conceive in my Mind the very Top of human Subtilty, and the Perfection of the whole Mathematical Sciences. His wonderful Inventions have been deliver'd to us by *Polybius*, *Plutarch*, *Tzetzes*, and others. *Conon* was Contemporary to *Archimedes*, one who was both a Geometrician and an Aſtronomer, whoſe Death *Archimedes* laments in his Book of the Quadrature of the *Parabola*. There followed *Archimedes* and *Conon*, and that at no great Diſtance, *Apollonius* of *Perga*, another Prince in Geometry, who was called, by way of high Encomium, *The Great Geometrician.* There are extant Four [now Seven] moſt ſubtle Books of his Conicks. To the ſame Perſon are aſcribed the 14 and 15 Books of *Euclid*, which were contracted by *Hypſicles*. *Hipparchus* and *Menelaus* wrote, the latter ſix, the other twelve Books of Subtenſes in a Circle; for which Invention, ſo very profitable and neceſſary, great Commendations and Thanks are due to both. There are alſo extant three Books of *Menelaus* concerning ſpherical Triangles. Three moſt uſeful Books of Sphericks of *Theodoſius* the *Tripolite* are alſo in the Hands of all. And theſe indeed, if you except *Menelaus*, lived all of them before Chriſt.

In the Year after Chriſt 70, there appeared in the World *Claudius Ptolemæus*, the Prince of Aſtronomers, a Man certainly wonderful, and (as *Pliny* ſaith) above the Nature of Mortals.

But he was not only moſt ſkilful in Aſtronomy, but in Geometry alſo; which as many other things written by him do witneſs, ſo eſpecially do the Books of Subtenſes: Thoſe of *Menelaus*, which were Six, and the Twelve of *Hipparchus*, all contracted by him into five Theorems. As for *Plutarch*, a moſt fam'd Philoſopher, there are extant his Mathematical Problems. And all know of the learned Commentaries of *Eutocius* the *Aſcalonite* upon *Archimedes*. By him are recited the Inventions of *Philo*, *Diocles*, *Nicomedes*, *Sphorus*, *Heron*, as of ſo many excellent Maſters in the Mathematicks, concerning doubling the Cube. *Heron*'s Genius certainly was excellent, as well for Mechanicks as Geometry. The Doubling of the Cube deliver'd by him is commended by *Pappus*, Book 3. *Prop.* 7, before all others. The admirable Works of *Cteſibius* the *Alexandrian*, to whom we owe our Pumps, are celebrated by *Vitruvius*, *Proclus*, *Pliny*, and *Athenæus*. The Name alſo of *Geminus* is not in the loweſt Place amongſt Mathematicians, whom *Proclus* has preferr'd in many things before *Euclid* himſelf.

Diophantus, and he alſo an *Alexandrian*, was as great in Arithmetic, as *Archimedes*, *Apollonius*, or *Euclid* in Geometry. He was certainly a Maſter of all Subtilty relating to Numbers. By him was found out that admirable Art which we call *Algebra*, which in theſe Times has been render'd more perfect and univerſal by *Francis Vieta* and *Renatus Carteſius*. There are others who are celebrated amongſt the Ancients alſo; as *Nicomachus*, famous for Arithmetical, Geometrical, and Muſical Monuments;

Serenus, well known to Geometricians for his two Books concerning the Section of a Cylinder; *Proclus*, *Pappus*, *Theon*. How great a Mathematician *Proclus* was, is manifeſt from his learned Commentaries on *Euclid*, and other Writings. And this is he, I ſuppoſe, who, as *Zonarus* reports, and from him *Ramus* and *Baronius*, about the Year of Chriſt 514, with Optic Artifice, and the Glaſſes which he us'd, burnt the Fleet of *Vitalian*, who was beſieging *Conſtantinople*. The Praiſes of *Theon*, which truly are deſervedly great, *Peter Ramus* wonderfully exaggerates, inſomuch that even the Books which hitherto all have aſcribed to *Euclid* ought, as he thinks, to be attributed to *Theon*. But *Ramus*, who every-where is ready to detract from *Euclid*, and this without grounding himſelf upon any ſolid Foundation, is not to be hearken'd to here. To come at length to a Concluſion: Let *Pappus* bring up the Rear, the laſt in Time amongſt the Ancients, as being one who liv'd about the Year 400; but in Reputation and all Mathematical Commendation to be reckon'd amongſt the firſt. *Alexandria*, that City ſo fruitful of Great Men, which before had brought forth *Hypſicles*, *Cteſibius*, and *Diophantus*, produced him alſo to the great Advantage of the Mathematicks. He wrote ſeven Books of Mathematical Collections, of which the two firſt are loſt. The five others do abound with ſo many, and ſuch various moſt noble Inventions in almoſt all Parts of the Mathematicks, that they are eſteemed amongſt the chief Monuments of the Ancients which are extant.

And thus you have a ſhort Hiſtory of the Origin and Progreſs of the Mathematicks. From which appears the Antiquity, Excellency, and Dignity of this Science. Certainly the ſame eminent Perſons in the Commonwealth of Learning, who diſcover'd Philoſophy, diſcover'd alſo the Mathematicks; like two Siſters born at one Birth, whom if any one would violently ſeparate from each other, he certainly attempts to break off their native Concord, with moſt notable Injury and, as it were, Cruelty to both; ſeeing, as it is wont to fall out in the Caſe of Twins, where they are remov'd from one another in Place or by Death; ſo it will be like to happen here, that, Mathematicks being plucked away from her, Philoſophy muſt needs languiſh and pine away.

Dr BAR-

Dr. BARROW's Words, prefixed before his *Apollonius.*

God always acts Geometrically.

HOW great a Geometrician art thou, O Lord? For while this Science has no Bounds; while there is for ever Room for the Diſcovery of New Theorems, even by Human Faculties, Thou art acquainted with them all at one View, without any Train of Conſequences, without any weariſome Application of Demonſtrations. In other Arts and Sciences our Underſtanding is able to do almoſt nothing; and like the Imagination of Brutes ſeems only to dream of ſome uncertain Propoſitions: Whence it is that in ſo many Men are almoſt ſo many Minds. But in theſe Geometrical Theorems all Men are agreed: In theſe the Human Faculties appear to have ſome real Abilities, and thoſe great, wonderful, and amazing. For thoſe Faculties, which ſeem of almoſt no Force in other Matters, in this Science appear to be efficacious, powerful, and ſucceſsful, &c. *Thee therefore do I take hence occaſion to love, and rejoice in, and admire; and to pant after that Day, with the earneſt Breathings of my Soul, when thou ſhalt be pleaſed, out of* *thy Bounty, out of thy Immenſe and Sacred Benignity, to grant me the Favour to perceive, and that with a pure Mind, and clear Viſion, not only theſe Truths, but thoſe alſo which are more numerous, and more important; and all this without that continual and painful Application of the Imagination which we diſcover theſe withal,* &c.

Mathematical Notes or Abbreviations.

= The Note for Equality. So $a=b$ ſignifies that a and b are equal.

+ The Note for Addition. So $a+b$ ſignifies the Sum of a and b together.

— The Note for Subtraction. So $a-b$ ſignifies the Difference between a and b.

× The Note for Multiplication. So $a\times b$ or ab ſignifies a multiplied by b.

:: The Note for Equality of Proportion. So $A:B::a:b$ ſignifies that A bears the ſame Proportion to B, that a bears to b.

∺ The Note of continued Proportion. So $A.B.C$ ∺ ſignifies that A bears the ſame Proportion to B, that B bears to C.

q The Note for a Square. So $CB.q.$ ſignifies the Square of the Line CB.

c The Note for a Cube. So $CB.c.$ ſignifies the Cube of the Line CB.

THE

THE ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

1. A Point is a Mark in Magnitude, which is [ſuppoſed to be] indiviſible.

That is, which cannot be divided ſo much as in Thought. A Point is the Beginning, as it were, of all Magnitude, as Unity is of Number.

2. A Line is a Magnitude which hath Length only, and wants all Breadth; foraſmuch as it is underſtood to be produced from the Flowing of a Point.

3. Points are the Terms of a Line.

Fig. 1. Tab. 1. 4. A right Line, is that which lies evenly betwixt its Terms.

Or, as *Archimedes:* A right Line is the leaſt of all thoſe which have the ſame Terms; or, is the ſhorteſt of all thoſe which can be drawn betwixt two Points.

Or, as *Plato* hath it: A right Line is that whoſe Extremes hide all the reſt: [That is, when the Eye is placed in a Continuation of the Line.]

The Senſe is the ſame in all. The Inſtrument whereby right Lines are deſcribed, is [called] a Rule; which whether it be ſtrait or not, you may know by this Trial:

Deſcribe a Line according to the Rule; then turning the Rule ſo that that which before was the Right-hand End may now become the Left-hand End, apply it again to the Line before deſcribed; if it doth now entirely fall in with the Line, the Rule is ſtrait; if not, the Rule is not ſtrait. The Reaſon hereof depends on Axiom 13.

5. A Surface is a Magnitude which hath only Length and Breadth.

It hath two Dimenſions therefore; and is underſtood to be produced by the Flowing of a Line.

6. Lines are the Extremes of a Surface.

7. A Plane, or a plain Surface, is that which lies evenly betwixt its extreme Lines.

Or, as *Hero,* that, to all the Parts whereof a right Line may be accommodated.

For it is produced from the Motion of a right Line.

Or, A plain Surface is that whoſe Extremes any of them hide all the reſt, [the Eye being placed in a Continuation of the Surface.]

Or, It is the leaſt of all Surfaces which have the ſame Terms. The Senſe is the ſame in all.

Euclid hath not here defined a Body or Solid, becauſe he was not yet about to treat concerning it. But, leſt any one ſhould want the Definition thereof, take it here thus, A Body is a Magnitude long, broad, and deep. A Body therefore hath three Dimenſions, a Surface two, a Line one, a Point none.

8. A plain Angle is the mutual Inclination to each other of two Lines, which touch one another in a Plain, and ſo as not to make one Line.

Therefore the two Lines BA, CA touching one another in A, but ſo as not to make one Line, conſtitute an Angle. Fig. 2, 4.

9. The Sides or Legs of an Angle are the Lines which make the Angle.

10. The *Vertex* or Top of an Angle is the Point (A) in which the Legs do meet and touch one another.

Note, That a ſingle Angle is deſigned by one Letter put at the Top: When there are more at one Point, they are deſigned by three Letters, the middlemoſt of which denotes the Top of the Angle; and many times alſo by one Letter interpos'd betwixt the Sides near the Top. So in Fig. 5. the Angle made by the Lines BA, CA is deſigned either by three Letters BAC, or by one only O.

11. Angles are called Equal, if when the Tops of them are laid upon one another, the Sides of one agree with the Sides of the other. But unto this it is not required that the Sides ſhould be of an equal Length.

12. They are called Unequal, when the Top and one Side agreeing, the other doth not agree;

and that is called the Greater, whoſe Side falls without. So the Angle BAE is greater than the Angle BAC. Fig. 5.

An Angle is not diminiſh'd or increas'd by the Diminution or Augmentation of the Sides that include it.

Fig. 2, 4. 13. A right-lin'd Angle is that which right Lines conſtitute; a curvi-lineal, which crooked Lines; a mix'd one, that which a right Line and a crooked one make.

Fig. 6. 14. When the right Line [CA] ſtanding upon the right one [BF] leans unto neither Part, and therefore makes the Angles on both Sides equal, CAB = CAF both of the equal Angles are called Right ones: But the right Line CA which ſtands upon the other, is called a perpendicular Line, or barely a Perpendicular.

Fig. 6. A right Angle may alſo be defined thus:

A right Angle is that that (BAC) when on the other Side an equal one ariſeth (CAF) if you produce or draw forth a Side, as (BA).

Two Rules ſo joined as to contain a right Angle, make an Inſtrument, which is called a Square. *Pythagoras* was the Inventor of it, as *Vitruvius* affirmeth, *c.* 2. *l.* 9. So great is the Uſe and Force of a right Angle in Framing, Meaſuring, and Strengthening all things, that nothing almoſt can be done without it. The Proof of a Square is made thus: Apply the Side of it, AE to the right Line AF, and deſcribe the right Line CA along the other Side. Then turning the Square towards B, if on both Sides it agrees to the right Lines CA, AB, you may know that it is true and exact. The Reaſon hereof appears from the 14th Definition itſelf.

Fig. 7. 15. The Angle BAC, which is greater than the right one FAC, is called an obtuſe Angle.

16. The Angle (LAI) which is leſs than the right Angle (FAI) is called an Acute one. Fig. 8.

17. A plain Figure is a plain Surface, bounded on every Side with one or more Lines.

18. A Circle is a plain Surface contained within the Compaſs of one Line called the Circumference; from which Line all the right Lines that can be drawn unto one certain Point, within the contained Space (A), are equal. Fig. 9.

19. That Point is called the Centre.

20. The Diameter is a right Line (BA) drawn through the Centre, and on both Sides ended at the Circumference; and conſequently it divides the Circle into two equal Parts. (As is abundantly manifeſt from the exact Agreement of two Semi-circles when laid one upon another.) Fig. 9.

21. The Semi-diameter or *Radius* is the right Line AF drawn from the Centre to the Circumference.

22. A Semi-circle is a Figure (BLC) which is contain'd by the Diameter BC, and half the Circumference (BLC.)

Mathematicians are wont to divide the Circumference into 360 equal Parts (which they call Degrees) the Semi-circumference into 180, the Quadrant or Quarter into 90.

23. A right-lin'd Figure is a plain Surface bounded on every Side with right Lines.

24. A Triangle is a plain Surface contained by three right Lines. Fig. 10.

This is the firſt and moſt ſimple of all right-lin'd Figures, and that into which they are all reſolv'd.

25. An equilateral Triangle is that which hath all the Sides equal. Fig. 10.

Fig. 11, 12. 26. An *Iſoſceles* or equicrural Triangle is that which hath only two Sides equal.

Fig. 13. 27. A *Scalenum* is that which hath three unequal Sides.

Fig. 13. 28. A right-angled Triangle is that which hath one Angle right.

Fig. 12. 29. An obtuſe-angled Triangle is that which hath one obtuſe Angle.

Fig. 10, 11. 30. An acute-angled Triangle is that which hath three acute Angles.

Fig. 14, 15. 31. Amongſt quadrilateral Figures, the Rectangle is that which hath four right, and conſequently equal Angles, whether the Sides be equal or not.

Fig. 15. 32. A Square is that which hath equal Sides, and is right-angled, and conſequently equiangled.

Every Square is a Rectangle; but every Rectangle is not a Square.

Fig. 16. 33. A Rhombus is a quadrilateral or fourſided Figure, which is equilateral, but not equiangled.

Fig. 17. 34. A Rhomboides is that which hath the oppoſite Sides and Angles equal; but is neither equilateral, nor equiangled.

Fig. 14, 15, 16, 17. 35. A Parallelogram is a quadrilateral Figure, which hath each two of its oppoſite Sides (AB, FC, and BF, AC) parallel to each other. Now what parallel Lines are will be ſhewed in the following Definition.

Every Rectangle and Square is a Parallelogram; but every Parallelogram is not a Rectangle or a Square.

Fig. 18. 36. Right Lines are Parallel or Equi-diſtant, which being in the ſame Plain, and drawn out on both Sides infinitely, are diſtant from one another by equal Intervals.

The Intervals are ſaid to be equal, in reſpect of the Perpendiculars. Wherefore if all the Perpendiculars (Q L) unto one of the two Parallels (A B) ſhall be equal, the right Lines (A B, C F) are ſaid to be Parallel.

Parallels are produced, if the right Line (L Q) which is perpendicular to the right Line (A B) be moved along (A B) always perpendicularly; for then its Extremity L deſcribes the Parallel C F.

37. The Diameter or Diagonal of a Parallelogram, and every Quadrilateral, is a right Line (A F) drawn through the oppoſite Angles. Fig. 17.

38. Plain Figures contain'd by more Sides than four, are called Many-ſided or Many-angled, and by a *Greek* Word *Polygones*.

39. The external Angle of a right-lin'd Figure, is that which ariſeth without the Figure when the Side is produced. Such are F B C, G C A, H A B. Every Figure therefore hath ſo many external Angles, as it hath Sides, and internal Angles. Fig. 19.

Poſtulates.

A Poſtulate is that which it is manifeſt in itſelf, that it may eaſily be done, or conceiv'd to be done. It is required therefore to be granted that we may,

1. From any Point given draw a right Line unto any other Point given.

2. Draw forth a finite right Line in Length ſtill farther.

3. From any Centre at any Interval deſcribe a Circle.

Axioms.

AN Axiom is a Truth manifeſt of itſelf.

1. Thoſe things which are equal to the ſame thing, are equal alſo amongſt themſelves. And that which is greater or leſſer than one of the Equals, is alſo greater or leſs than the other of them.

2. If to Equals you add Equals, the Wholes will be equal.

3. If from Equals you take away Equals, the Remainders will be equal.

4. If to Unequals you add Equals, the Wholes will be unequal.

5. If from Unequals you take away Equals, the Remainders will be unequal.

6. What things are each of them half of the ſame Quantity, are equal amongſt themſelves; and what things are double, or treble, or quadruple of the ſame, are equal amongſt themſelves.

7. What things do mutually agree with one another are equal.

8. If right Lines be equal, they will mutually agree with one another; and the ſame thing is true of Angles.

9. The Whole is greater than its Part.

10. All right Angles are equal amongſt themſelves.

11. Parallel Lines have a common Perpendicular: That is, the right Line which is perpendicular to one of them is perpendicular alſo to the other.

Fig. 21. 12. The two perpendicular Lines (LO, QI) intercept equal Parts of the Parallels.

13. Two right Lines do not comprehend a Space; for unto this there are required three at the leaſt.

14. Two right Lines cannot have one common Segment; for that they cut one another only in a Point.

Of Propoſitions ſome propoſe ſomething to be done and are called Problems; in others we proceed no further than bare Contemplation, which thereſore are named Theorems.

PROPOSITIONS.

THE requiſite Citations are ſound in the Margin. When Propoſitions are cited, the firſt Number deſigns the Propoſition; the Letter *l*, with the Number following, ſignifies the Book. As when you meet with *(per* 5. *l* 3.) you muſt read it thus, (by the 5th Propoſition of the 3d Book). The Figure is always to be ſought amongſt the Figures of that Book in which we are then converſant. The reſt of the Citations are eaſy to be underſtood.

The primary Affections of Triangles and Parallelograms are delivered in this firſt Book: The more famous Propoſitions are, 32, 35, 37, 41, 44, 45, 47.

PROP. I. Problem.

UPON a given Right Line (AB) to make an Equilateral Triangle. Fig. 23.

From the Centre A, with the Interval (AB) [a] deſcribe the Circle FCB: and from the Centre B with the ſame Interval BA deſcribe the Circle ACL, cutting the former in the Point

[a] *Per Poſtul.* 3.

Point C, from which Point draw the right Lines CA, CB.

I ſay, that the Triangle ACB now made, is Equilateral. For the right Line AC is equal to the right [b] Line AB, ſeeing they are Semi-diameters of the ſame Circle FCB: And again, the right Line BC is equal to the ſame right Line BA, ſeeing they are both Semi-diameters of the Circle LCA. Therefore AC, BC are [c] equal betwixt themſelves. And therefore all the Sides of the Triangle are equal. Therefore the Triangle [d] ACB is both Equilateral, and made upon the given Line AB; which was the thing to be done. *Q. E. F.*

[b] *Per Def.* 18.

[c] *Per Axiom* 1.

[d] *Per Def.* 25.

Fig. 77. Corollary. *Hence we may meaſure an inacceſſible Line, as AB. For ſuppoſe any Equilateral Triangle whatſoever BDE applied to the Point B along the Line BA. Looking from the Point B along the Line BE, mark as many Points as you conveniently can in the Line BC. Then remove the Triangle BDE along the Line BC, from one place to another of that Line, until, by taking aim along the Side of the Triangle ED or CF, you ſee the inacceſſible Point A in a Continuation of that Line. Thus the Triangle BAC is as well Equilateral as BDE. If therefore you ſhall now meaſure the acceſſible Line BC, you have the Meaſure of the inacceſſible AB. Q. E. I.*

PROP. II. Problem.

Fig. 24. F*ROM a given Point A to draw a Right Line equal to one given EF.*

Take with a Pair of Compaſſes the Interval EF

E F and transfer it from A to D, the right Line A D will be equal to the given E F.

PROP. III. Problem.

Fig. 24.

TWO unequal Right Lines being given, from the greater of them G H *to cut off* G I *equal to the leſs* E F.

Take with a Pair of Compaſſes the Interval of the leſſer given Line E F, and transfer it unto the greater from G to I.

PROP. IV. Theorem.

Fig. 25.

IF in two Triangles (X, Z) one Side of the one (B A) be equal to one Side F L *of the other, and another Side (C A) of the one equal to another Side (I L) of the other, and the Angles (A and L) made by thoſe Sides be alſo equal; then the Baſes (B C, F I) are likewiſe equal, as alſo the Angles at the Baſes (B, F, and C, I) which are oppoſite to equal Sides, and conſequently the whole Triangles are equal.*

For if we ſuppoſe the Triangle Z to be laid upon the Triangle X, the Sides L F, L I, will perfectly agree and fall in together with the Sides that are equal to them, A B, A C, and this in ſuch ſort [c] that the three Points (L, F, I) ſhall fall upon the three Points (A, B, C). Therefore the whole Baſe F I will alſo fall upon the whole Baſe B C. But then the Angles F, B, and likewiſe thoſe I, C and the whole Triangles will mutually *(congruere)* agree to each

[c] *Per Axiom* 8.

each other. All therefore, by Axiom 7th, are equal. *Q. E. D.* Which was the Thing to be demonſtrated.

Fig. 78. Coroll. (1.) *Hence we may alſo in another way meaſure the inacceſſible Line AB. For from any Point whatſoever, as the Point C, let the Angle ACB be obſerved, and then let the Lines AC, BC be meaſured: and in any acceſſible Plain let there be meaſured about the Angle F, which is equal to the Angle C, two Lines FD and FE which are equal to the Lines AC and BC reſpectively. And then there will be the acceſſible Line DE equal to the inacceſſible AB. Q. E. I.*

Fig. 79. Coroll. (2.) *Hence alſo, thoſe who play at Billiards with Ivory Balls may learn how by the Reflexion of their own to hit and remove their Adverſaries Ball. For let B be the Ball to be ſtricken, A that which is to ſtrike it, and CD the Rectilinear Plain. Let the Line BE be perpendicular to the Line CD, and DE be equal to DB. If the Ball A be ſtricken and carried along the right Line AFE unto the Point F, it will there be ſo reflected, that after the Reflection it will tend unto B. For in the Triangles BFD, EFD, the Side FD is common to both, and the Side DB is equal to the Side DE; and the Angles at D are equal as being right ones. The whole Triangles therefore are equal: and therefore the Angle BFD, which is equal to the Angle DFE, is* [d] *equal to AFC, the Angle AFC being vertically oppoſite to DFE. Wherefore, ſeeing the Angle AFC is the Angle of Incidence, which in ſuch Caſes is equal to the Angle of Reflexion, it is manifeſt that BFD, which hath been proved equal to AFC, is the Angle of the Reflexion of the Ball A, and that*

[d] *Per* 15. *l.* 1.

that the Ball tending towards E is in the Point F ſo reflected as to hit the Ball B. Q. E. D.

Scholium or Obſervation.

BY much the ſame Way of Reaſoning, whereby this 4th Propoſition has been demonſtrated, the following Theorem, which we ſhall have occaſion to uſe by and by, may be demonſtrated alſo.

If in two Triangles X, Z, the Sides B C and F I ſhall be equal, and the Angles adjacent to theſe two Sides equal alſo, *viz* B and C equal to F and I: All the other Things, and the whole Triangles themſelves will be equal. Fig. 25.

For the Side F I laid upon the Side B C will agree, or thoroughly coincide with it [e]. And then becauſe the Angles B and C are equal to thoſe F and I, when the Side F I is laid upon the Side B C, F L [f] will fall exactly upon B A, and I L upon C A. Therefore the Point L will fall upon the Point A (for if it fall without A, the Sides F L, I L would not fall upon the Sides B A, C A). Therefore all Things are equal by Axiom 7th.

[e] *Per Axiom* 8.

[f] *Per Axiom* 8.

PROP. V. Theorem.

IN an Iſoſceles or Equicrural Triangle, the Angles at the Baſe (A, C) are equal. Fig. 26.

Let the Triangle ABC be underſtood to be twice put, but in an inverted Poſture *cba*. Becauſe therefore, in the two Triangles A B C, *cba* the Side A B is by the Suppoſition equal to the Side *cb*, and the Side CB to the Side *a b*, and the Angle B to the Angle *b*; the Angle A alſo at the Baſe will [g] be equal to the Angle *c*. Q. E. D. For as for the Angles C and *c* they are the ſame.

[g] *Per* 4. *l.* 1.

Corollary.

THerefore an Equilateral Triangle is alſo Equiangular.

PROP. VI. Theorem.

Fig. 26. *IF in a Triangle (ABC) two Angles (A and C) be equal, the Sides alſo (AB, BC) which are oppoſite to thoſe Angles are equal alſo.*

Let the Triangle ABC be ſuppoſed to be twice put, but in an inverſe Situation, *c b a*. Becauſe therefore in the Triangles A B C, *c b a*, one Side A C is equal to one Side (*c a*), and the Angle A is equal to the Angle *c*, and the Angle C equal to the Angle *a*, all the other things ſhall be likewiſe [h] equal, and conſequently AB ſhall be equal to the Side *c b*. *Q. E. D.* For as for the Lines CB and *c b* they are the ſame.

[h] *Per Schol. Prop.* 4.

Corollary.

THerefore an equi-angled Triangle is alſo equilateral.

Fig. 80. Coroll. (2.) *Hence, by the means of the Shadow of the Sun, we may meaſure the Height of a Tower, or any elevated Point. For when the Sun is elevated* 45 *Degrees above the Horizon, the Shadows*

Shadows which the Towers caſt towards the Horizon will be exactly equal to their Height. For, by reaſon that the Angle ACB is half a right Angle, the Angle BAC alſo[i] *will be half a right one; and, ſo by the Force of the preſent Propoſition, the Line AB will be equal to the Line BC. The Line BC therefore being found by meaſuring, there is found at the ſame time the Line AB, the Height of the Tower above the Horizon.*

[i] *Per Corol.* 11. *Prop.* 32. *l.* 1.

Coroll. (3.) *The ſame thing alſo may be found without the Sun by the means of an aſtronomical Quadrant. For where the Angle of Elevation is half-right, there the Height of the Tower above the Obſerver's Eye is equal to the Diſtance of the ſame Eye, from that Part of the Tower which is oppoſite to it. The Diſtance therefore of the Eye from the Tower being given by meaſuring, there is given at the ſame time the Height of the Tower. Q. E. I.*

The VIIth Propoſition in *Euclid* is for the ſake of the VIIIth, which without it will here be demonſtrated.

PROP. VIII. Theorem.

Fig. 27.

If two Triangles (X, Z) have all their Sides equal amongſt themſelves reſpectively (AC equal to EF, CB to FI, AB to EI) they will alſo have all the Angles, which are oppoſite to equal Sides, equal (C equal to F, A to E, B to I).

For ſuppoſe the Side AB laid upon its Equal EI, if then the Point C falls upon F, the Triangles will in the whole agree or coincide, and conſequently all the Angles will be equal. But the Point C will fall upon the Point F. For,

Fig. 81. *From the Centre A let a Circle be deſcribed with the Semidiameter E F; and from the Centre I let another Circle be deſcribed with the Semidiameter I F; the Point C, by reaſon of the Equality of the Sides of both Triangles, will be in the Circumference of both Circles, and conſequently in the Point E, the common Interſection of both theſe Circumferences. Q. E. D.*

PROP. IX. Problem.

Fig. 29. *TO Biſect or Divide into two equal Parts a given right-lin'd Angle, as IAL.*

From the Sides of the Angle take with a Pair of Compaſſes two equal Lines, A B, A C; then from the Centres B and C deſcribe two equal Circles cutting one another in F; which done draw the Line F A. This biſects the Angle.

For draw the Lines BF, CF; the Triangles FAB, FAC are to each other Equilateral; for the Sides A B, A C are by the Conſtruction equal, as in like manner are the Sides BF, CF, they being Semidiameters of equal Circles; and A F is common to both Triangles. Therefore the Angles BAF, CAF [k] are equal. Therefore the given Angle I A L is biſected. Q. E. F.

[k] *Per* 8. *l.* 1.

Corollary.

HENCE we may learn how an Angle may be divided into equal Angles, 4, 8, 16, *&c. viz.* by biſecting each Part again.

Scholium.

Scholium.

NO one hath hitherto taught the Way of dividing Angles into all equal Parts whatſoever with a Pair of Compaſſes, and a Rule.

Yet may you divide any given Angle mechanically into any equal Parts whatſoever, if from the Top of the Angle as the Centre you deſcribe an Arch between the Legs of the Angle, and divide that Arch into as many equal Parts as you require; for right Lines let down from A through the Points of the Diviſion will cut the Angle into ſo many equal Parts. Fig. 30.

PROP. X. Problem.

T*O biſect a finite given Line (AB).* Fig. 31.

Upon the given A B make an Equilateral [l] Triangle A G B. Biſect its Angle G[m] with the right Line G C. The ſame ſhall biſect the given Line A B.

For in the Triangles X, Z the Side C G is common; and by the Conſtruction G B, G A are equal, and the Angles contained between them A G C, B G C, are likewiſe equal. Therefore the Baſes A C, B C[n] are equal. The given Line therefore A B is biſected. *Q. E. F.*

But for Practice it is ſufficient from the Centres A and B to deſcribe two equal Circles, cutting one another in G and L, and ſo to draw the right Line G L.

[l] *Per* 1. *l.* 1.

[m] *Per præced.*

[n] *Per* 4. *l.* 1.

PROP. XI. Problem.

Fig. 32. F*ROM a given Point (A) to raiſe a Perpendicular in a given right Line (L I).*

With a Pair of Compaſſes take the equal Lines A C, A F. From the Centre C and F deſcribe two Circles, cutting one another in B. The Line which is drawn from B to A will be the Perpendicular required.

For let the right Lines C B, F B be drawn. The Triangles X and Z are equilateral to one another. Therefore the Angles C A B, F A B are equal [o]. Therefore B A is [p] perpendicular to the Line (L I). *Q. E. F.*

In Practice this and the next are eaſily performed by the Help of a Square.

[o] *Per* 8. *l.* 1.

[p] *Per Def.* 14.

PROP. XII. Problem.

Fig. 33. F*ROM a given Point (A) which is without an infinite right Line (as L Q) to let fall a Perpendicular to that Line.*

From the Centre A deſcribe a Circle which may cut the given L Q in C and I. Biſect the right Line C I [q] with the right Line A B. This A B is the Perpendicular required.

For let there be drawn A C, A I. Becauſe by the Conſtruction X and Z are equilateral to one another; therefore the Angles [r] C B A, I B A are equal. Therefore A B is [s] perpendicular. *Q. E. F.*

[q] *Per* 10. *l.* 1.

[r] *Per* 8. *l.* 1.

[s] *Per Def.* 14.

PROP.

PROP. XIII. Theorem.

THE right Line (BA) ſtanding upon the right Line (CF) either makes two right Angles, or Angles equal to two right ones. Fig. 34.

For if BA ſtand upon it perpendicularly, then by Definition 14 the two Angles BAC, BAF will be right ones. And if BA ſtand obliquely, let there be rais'd [t] the Perpendicular AL. Where becauſe the unequal Angles CAB, FAB poſſeſs the ſame Place which the two right ones CAL, LAF do, and agree to them, they are equal [u] to them. *Q. E. D.*

[t] *Per* 11. 1.

[u] *Per Axiom* 7.

Corollaries.

1. IN the ſame manner it will be demonſtrated, if more right Lines than one ſtand upon the ſame right Line, that the Angles thereby made are equal to two right ones.

2. Two right Lines, cutting one another, make the Angles equal to four right ones. Fig. 37.

3. All the Angles, which are about one Point, make Angles equal to four right ones. It appears from Corollary 2. Fig. 36.

4. The Angle CAF being known, you at the ſame time know its Complement unto two right Angles BAF. For Example, let the Angle CAF be of 70 Degrees; the Angle BAF will be of 110 Degrees. For thoſe two Numbers added together make 180 Degrees, which is the Meaſure of two right Angles. Fig. 37.

PROP. XIV. Theorem.

Fig. 35.

IF two right Lines (XR, ZR) at the ſame Point of a right Line QR make the Angles on both Sides (XRQ, ZRQ) equal to two right Angles; the Lines (XR, ZR) make one right Line.

If you deny it, let XR, BR make one right Line. Therefore the Angles XRQ, QRB[x] will make two right Angles. Which thing is[y] abſurd: ſeeing by the Hypotheſis XRQ, ZRQ do make two right Angles.

[x] Per 13. 1.

[y] Contra Axiom 9.

PROP. XV. Theorem.

Fig. 37.

IF two right Lines (BC, FL) cut one another in A, the Angles oppoſite at the Top (A) are equal, viz. LAB to CAF, and BAF to LAC.

For becauſe BA ſtands upon the right Line LF, the Angles LAB, FAB are[z] equal to two right ones: And becauſe FA ſtands upon the right Line BC, the Angles FAC, FAB are alſo equal[a] to two right ones. Therefore the two Angles together[b] LAB, FAB are equal to thoſe two together CAF, FAB. Therefore taking away the common Angle, FAB, LAB[c] remains equal to CAF. In the ſame manner BAF, LAC are ſhewed to be equal.

[z] Per 13. 1.

[a] By the ſame Prop.

[b] Per Axiom 1.

[c] Per Axiom 3.

Corollary. *From theſe two Propoſitions we gather in Catoptrics that a Ray of Light, as reflected in an Angle equal to the Angle of Incidence,* *taketh the ſhorteſt Way of all.* E. G. *When the Angles BED, AEF are equal, the Lines AE and EB, taken together, are ſhorter than any Lines whatſoever, as AF and FB taken together. For from the Point B let the Perpendicular Line BC be let down; and let BD and DC be equal; let the Lines alſo EC and FC be drawn. Now in the Triangles BED and DEC, ſeeing the Side DE is common to both, and the Side BD and DC are equal by the Hypotheſis, as is alſo in the like manner BDE equal to the Angle CDE; the Triangles ſhall be*[a] *equal in all other things, and BE ſhall be equal to CE, and the Angle BED to the Angle DEC: (where becauſe the Angle DEC is equal to [BED, that is] AEF, the Lines AE, EC are prov'd to make one right Line.) And in the ſame manner the Line BF will be proved equal to FC. Seeing therefore the Lines BE and EA, taken together, are equal to the Line CA, and the Lines BF, FA taken together are equal to the Lines CF, FA taken together; it is manifeſt that CA, which is one Side of the Triangle ACF*[e]*, is leſs than the two Sides CF, FA taken together.* Q. E. D. Fig. 82.

[a] *Per* 4. *l.* 1.

[e] *Per* 20. *l.* 1.

PROP. XVI, XVII.

T*HESE two* Propoſitions *are contain'd in* Propoſition 32. *and are not here made uſe of till then.*

PROP. XVIII. Theorem.

I*N every Triangle the Angle (A) which is oppoſed to the greater Side (BO) is greater; and (B) which is oppoſite to the Side (AO) is the leſſer Angle.* Fig. 38.

(A) Cannot be equal to (B) for then the oppoſite Sides B O, A O would be equal[f]; which is contrary to the Hypotheſis. Neither can A be leſs than B, for if it were ſo, there might within the Angle B be made an Angle A B F by the right Line B F; which Angle ſhould be equal to A. But then by the 6th of this Book B F, A F ſhould be equal; and if you add to both O F, then B F, F O ſhall be equal to A O. But A O by the Hypotheſis is leſs than B O. Therefore B F, F O ſhall be leſs than B O, which contradicts the Definition of a right Line, which is the ſhorteſt of all betwixt two Points. Therefore the Angle A is neither leſs than B, nor equal to it. Therefore it is greater. *Q. E. D.*

[f] *Per* 6. *l.* 1.

PROP. XIX. Theorem.

Fig. 38.

IN the Triangle A O B the Side (B O) which is oppoſed to the greater Angle (A) is the greater; and (A O) which is oppoſed to the leſſer Angle B is the leſſer.

This Propoſition is the converſe of the former. B O is not leſs than A O, for if it were, the Angle (A) by the 18th would be leſs than B; which is contrary to the Hypotheſis. Nor can B O be equal to A O, for in this Caſe by the 5th the Angles A and B would be equal. But this Equality of thoſe Angles is contrary to the Hypotheſis. Therefore B O is greater than A O. *Q. E. D.*

Fig. 83.

Corollary. *Hence we gather that a Globe or Ball perfectly poliſhed cannot reſt in an horizontal Plane perfectly poliſhed, but where it touches the Earth.*

Earth. For let the Line A B be an horizontal Plane, C the Earth's Centre, C A the Semidiameter of the Earth, perpendicular to the Tangent AB. The Globe placed at B, becauſe of its Gravity, and the Declivity of the Plane, will deſcend towards A. For in the Triangle C A B the perpendicular Line C A, which is oppoſite to the acute Angle A B C, is leſs than the Line B C which is oppoſed to the right Angle B A C; and ſo there is from B to A a perpetual Deſcent, in which the Globe cannot reſt. And in the like manner we prove the Deſcent of Fluids, and their Conformation into a ſpherical Surface.

PROP. XX. Theorem.

IN any Triangle, any two Sides of it, taken together, are greater than the remaining Side.

This with *Archimedes* is as it were an Axiom; foraſmuch as it is immediately manifeſt out of his Definition of a right Line; which ſee above amongſt the Definitions.

PROP. XXI. Theorem.

IF from the Ends of one Side AB two right Lines be drawn, and joined together within the Triangle, (as the Lines A O, B O); theſe are leſs than the Sides of the Triangle (AC, BC), but they comprehend a greater Angle (A O B). Fig. 39.

For as for the firſt Part of the Propoſition, Draw out A O unto F. A C, C F are[g] greater than A F. Therefore the common Line F B being added, A C, B C are greater than A F, F B. Again, O F, B F are greater[h] than O B. Therefore the common A O being added, A F, B F are greater than A O, B O. Therefore A C, C B are much greater than A O, O B.

[g] *Per* 20. *l.* 1.

[h] By the ſame.

The ſecond Part of this Propoſition will be demonſtrated in the ſecond Corollary of the firſt Part of Propoſition 32. And in the mean while we ſhall make no uſe of it.

PROP. XXII. Problem.

Fig. 40.

TO make a Triangle of three given right Lines (B O, L B, L O) (of which any two muſt be greater than the third).

Let B L one of the given Lines be taken, and B one of its Extremities being taken for the Centre, with the Interval of the other given Line B O deſcribe an Arch.

Then the other Extremity L being taken for the Centre, with the Interval of the third given Line L O deſcribe an Arch, cutting the former in O; which being done, and the right Lines B O, L O being drawn, I ſay that that is done which was to be done.

The Demonſtration is manifeſt from the Conſtruction.

PROP.

PROP. XXIII. Problem.

AT a given Point in a right Line (as B) to make an Angle equal to a given one (A). Fig. 40.

Firſt of all let C F be drawn at a venture, cutting the Sides of the given Angle A. Then in the given right Line from B take B L equal to A F. Then from the Centre B deſcribe a Circle with the Interval A C; afterwards another from the Centre L with the Interval F C, which may cut the former in O. Then from O unto B and L having drawn right Lines: The Angle L B O will be equal to the given one A.

For by the Conſtruction the Triangles are equilateral to one another. Therefore by the 8th of this Book the Angles B and A are equal.

Scholium.

IT ſeems meet for the ſake of Beginners to propound ſome things here which are neceſſary for Practice about Angles.

The Meaſure of an Angle is the Arch of a Circle, which is deſcribed from A, the Top of the Angle as the Centre. Therefore look how many Degrees the Arch B C which is intercepted between the Legs of the Angle B A C ſhall contain, of ſo many Degrees the Angle B A C ſhall be ſaid to be. And ſo becauſe B F, a Quarter of the Circumference, contains 90 Degrees, and meaſures the right Angle B A F, a right Angle ſhall be ſaid to be Fig. 41.

be of 90 Degrees. In like manner becauſe half the Circumference, which is divided into 180 Degrees, meaſures two right Angles, and the whole Circumference, which is divided into 360 Degrees, meaſures four right Angles; two right Angles ſhall be ſaid to make 180 Degrees, and four, 360 Degrees. Theſe Things being premiſed, the Practice about Angles is as follows:

Fig. 42. 1. At B a given Point in a right Line to make an Angle equal to the given one A.

From A the Top of the given Angle as the Centre, deſcribe betwixt the Sides the Arch CF. Then from B the given Point as the Centre deſcribe with the ſame Interval the Arch LZ; from which take off LO equal to CF. Through B and O draw a right Line; LBO ſhall be equal to the given A.

Fig. 43. 2. To examine the Degrees of the given Angle OPQ. This is done very eaſily thro' any Semicircle or Protractor, which is divided into 180 Degrees. For put the Centre of the Semicircle upon P the Top of the Angle, and the Radius of the Semicircle PL upon the Side of the Angle PQ; and the Arch LO which is intercepted betwixt the Legs of the Angle will ſhew of how many Degrees the given Angle is.

3. To frame an Angle containing a given Number of Degrees, as 42:

Fig. 43. Draw the right Line XQ, in which mark the Point P. Upon P put the Centre of the Semicircle, and its Semidiameter PL upon PQ. From L number 42 Degrees, that is, until you come to O. A right Line, drawn from P through O, will give the Angle OPL of 42 Degrees.

PROP.

PROP. XXIV. and XXV. Theorems.

IF two Triangles (B A C, B A F) ſhall have two Sides (B A, A C) equal to two (B A, A F) one Side of one, to one Side of the other; and if one of the Triangles hath the Angle (B A F) contained by thoſe Sides greater than the other (B A C); it ſhall have the Baſe B F greater than the Baſe (B C). Fig. 44.

And again, If it hath the Baſe greater, it ſhall have the Angle greater.

From the Centre A deſcribe a Circle which paſſeth through C, it ſhall paſs alſo through F, becauſe A C, A F are ſuppoſed to be equal. Therefore B F ſhall fall betwixt the Point A and C. Then join C F. The Angle B C F is greater than the Angle A C F; that is, by the 5th of this Book, than the Angle A F C, and conſequently much greater than the Angle B F C. Therefore in the Triangle B C F[i], B F which is oppoſite to the greater Angle B C F is greater than B C which is oppoſite to the leſſer Angle B F C.

[i] *Per* 19. *l.* 1.

2. As for the ſecond Part of the Propoſition, this is manifeſt from the firſt Part.

PROP. XXVI. Theorem.

IF two Triangles (X and Z) have two Angles equal to two, one Angle of the one equal to one Angle of the other (B to F and C to I), and one Side of one equal to one of the Fig. 25.

the other, whether it be that which is betwixt the equal Angles (as $BC = FI$*) or a Side which is oppoſed to one of the equal Angles (as* $AC = LI$*); all the other Parts ſhall be equal.*

For in the firſt Place let the Sides (B C, F I) which are betwixt the equal Angles be ſuppoſed equal; in this Caſe all the other Parts are equal; as hath been already demonſtrated in the *Scholium* of Propoſition 4.

Again, Suppoſe the Sides A C, L I which are oppoſed to the equal Angles to be equal. Here becauſe the Angles (B, C) are by the Hypotheſis equal to (F, I) the other Angles alſo (A, L) ſhall be equal by Coroll. 9. Prop. 32. which Propoſition depends not upon this. Therefore by the firſt Part of this all the other Parts are equal.

Fig. 84. Coroll. *Hence alſo, following* Thales, *we may meaſure inacceſſible Diſtances.* E. G. *Let A D be an inacceſſible Line: to which at the Point A let there be erected the Perpendicular A C. Let there be made the Angle (A C B) equal to the Angle (A C D) the acceſſible Line A B ſhall be equal to the inacceſſible A D.* Q. E. I.

PROP. XXVII. Theorem.

Fig. 45. *If the right Line G O ſhall cut two right Lines which are parallel (A B, C F); 1. The alternate Angles (R L O, Q O L, likewiſe B L O, C O L) ſhall be equal. 2. The external Angle G L B ſhall be equal to the internal one on the ſame Side*

(that is, to L O F): as likewiſe G L R equal to L O C. 3. The two internal ones on the ſame Side (A L O, C O L) as taken together, ſhall be equal to two right ones, as likewiſe the two (B L O, F O L) equal to two right ones.

The firſt Part is thus proved. From O and L draw the Perpendiculars O R, L Q. Theſe are perpendicular to the [k] two Parallels A B, C F; and by Definition 36. equal betwixt themſelves, they ſhall therefore [l] intercept equal Parts of the Parallels, and R L ſhall be equal to Q O. Therefore the Triangles X and Z are equilateral to one another. Therefore [m] the alternate Angles R L O, Q O L which are oppoſite to the equal Sides R O, Q L are equal. Which is the firſt thing. From whence it is likewiſe manifeſt that the Alternates B L O, C O L are equal. For becauſe as well B L O, A L O as C O L, F O L are [n] equal to two right ones; therefore B L O, A L O together are equal to C O L, F O L. Therefore taking away the Equals R L O, F O L, the remaining ones B L O, C O L ſhall be likewiſe equal.

Part ſecond. The Angle G L B is equal to that which is vertically oppoſite R L O [o]; but R L O by the firſt Part of this Propoſition is equal to L O F. Therefore G L B the external Angle is equal to the internal remote one which is on the ſame Side, L O F.

Part third. A L O by the firſt Part is equal to L O F. But L O F with C O L makes Angles equal to two right ones. Therefore A L O with C O L doth the ſame.

Corol-

Fig. 46.

[k] *Per Axiom* 11.

[l] *Per Axiom* 12.

[m] *Per* 8. *l.* 1.

[n] *Per* 13. *l.* 1.

[o] *Per* 15. *l.* 1.

Fig. 85. Corollary. *Hence in Imitation of* Eratoſthenes *we learn to meaſure the Compaſs of the Earth. For he obſerved that, on the Day of the Summer Solſtice, the Sun was perpendicularly over* Siene, *a City of* Egypt; *and he found by the means of a Style perpendicularly erected, that on the ſame Day the Sun was diſtant from the vertical Point of* Alexandria, *a City of* Egypt, *ſituate almoſt under the ſame Meridian with the other, ſeven Degrees, with one 5th Part of a Degree; and he knew that theſe two Cities were about 5000 Furlongs diſtant from each other. From theſe Things by the Help of this Propoſition he determin'd the Compaſs of the Earth. Let A be* Siene, *and B be* Alexandria, *where the Gnomon BC is erected perpendicular to the Horizon. Let DF and EG be the Solar Rays parallel to one another as to Senſe. DA a Ray perpendicular to the Horizon of* Siene; *and EG a Ray oblique to the Horizon of* Alexandria, *and which paſſing by the Top of the Gnomon makes with it the Angle GCF which is of $7\frac{1}{5}$ Degrees. Now ſeeing the Angle GCF is equal to the alternate one AFB, and the Meaſure of it is the Arch AB of $7\frac{1}{5}$ Degrees, he found the Compaſs of the Earth by this Analogy; as $7\frac{1}{5}$ Degrees are to 5000 Furlongs; ſo the whole Circumference, which is of 360 Degrees, is in a groſs Number to 250000, the Compaſs of the Earth in the ſame Meaſure.* Q. E. I.

PROP. XXVIII. Theorem.

Fig. 47. *IF a right Line (GO) cutting two right Lines (AB, CF) makes the alternate Angles (ALO, LOF) equal; the Lines (AB, FC) are parallel.*

If you deny it, let XLZ paſſing through the Point L be parallel to CF. Therefore XLO[p] is equal to the alternate FOL, which cannot be, ſeeing by the Hypotheſis ALO is equal to FOL.

[p] By the foregoing.

PROP. XXIX. Theorem.

Fig. 45, 46.

If a right Line GO cutting two right Lines (AB, CF) ſhall make the external Angle (GLB) equal to the internal oppoſite one (LOF), or ſhall make the two internal Angles on the ſame Side (ALO, COL) equal to two right Angles; (AB, CF) are parallel Lines.

By the 15th of this Book GLB is equal to ALO, which is vertically oppoſite to it. But by the Hypotheſis GLB is equal to LOF. Therefore alſo ALO is equal to its alternate one LOF. Therefore[q] AB, CF are parallel.

Again, COL with FOL makes Angles equal to two right ones. But by the Hypotheſis COL with ALO makes in all two right Angles alſo. Therefore ALO, FOL the alternate Angles are equal. Therefore, again[r], AB, CF are parallel.

[q] By the foregoing.

[r] Ibid.

Corollary. *From the ſecond Part of this Propoſition it appears that every Rectangle is a Parallelogram.*

PROP. XXX. Theorem.

Fig. 45.

If two right Lines (AB, CF) be parallel to the ſame right Line (DN) they are parallel betwixt themſelves.

It is manifeſt in itſelf, and from the foregoing Propoſitions. For if all be cut by the right Line G O, the external Angle G L B is equal[s] to the internal oppoſite one L D N. Now L D N is an external Angle in reſpect of D O F, and therefore[t] equal to it. Therefore alſo G L B is equal to L O F. Therefore A B, C F[u] are parallel.

[s] *Per* 27. *l.* 1.

[t] By the ſame.

[u] By the foregoing.

PROP. XXXI. Problem.

Fig. 48.

THROUGH a given Point (A) to draw a Parallel to a given right Line (C F).

From the Point A let there be drawn at random A L, cutting the given F C. At the Point A let there be made the Angle[x] L A S equal to the Angle A L F. The Line A S will be parallel to C F, as is manifeſt from the 28th, the alternate Angles S A L, A L F being equal.

[x] *Per* 23. *l.* 1.

As for the Practice: Draw A L, and from the Centre L deſcribe an Arch I Q; and from the Centre A with the ſame Interval deſcribe the Arch O X; from which having taken off O B equal to I Q; the right Line drawn thro' A and B will be the Parallel ſought. The Demonſtration depends upon the 29th l. 1.

Fig. 49.

Or otherwiſe thus: From a certain Centre P deſcribe a Circle which may paſs through the given Point A, and may cut the given Line C F in Q and O. Take the Arch O N equal to Q A. The right Line A N ſhall be the Parallel ſought.

The Demonſtration hereof depends upon 29. l. 3, and the 28th of this.

PROP.

PROP. XXXII. Theorem.

PART I.

IN every Triangle any one of the external Angles (as FBC*) is equal to the two internal remote ones (A and C).* Fig. 51.

Through the Point B draw[y] BL parallel to AC. Becauſe FA cuts the two Parallels BL, AC, the external Angle FBL ſhall be equal to the internal one A[z]. And becauſe the Line BC cuts the ſame Parallels (BL, AC); the Angle LBC ſhall be[a] equal to its alternate one C. Therefore the whole Angle FBC ſhall be equal to A and C both together. *Q. E. D.*

[y] *Per* 31. *l.* 1.

[z] *Per* 27. *l.* 1.

[a] By the ſame.

Corollaries.

1. THE external Angle FBC is greater than either of the internal oppoſite ones A or C. Fig. 51.

2. Of the Angles (C and AOB) having the ſame Baſe, AOB, which falls within, is the greater. Fig. 39.

For let AO be produced unto F, AOB by this Propoſition is greater than OFB; and likewiſe OFB is by this greater than C. Therefore AOB is much greater than C.

3. If from one Point A there fall two right Lines upon BC; one of them AO obliquely, the other AF perpendicularly; this laſt ſhall fall on the Side of the acute Angle AOB. For let it fall, if it may be, on the Side of the obtuſe Angle AOC, as for Inſtance in Q. In this Caſe the acute Angle AOB ſhall be exter- Fig. 55.

nal in reſpect of A Q B; and conſequently ſhall be greater than the right one, by *Coroll.* 1. which is abſurd.

PROP. XXXII. Theorem.

PART II.

IN every Triangle the three Angles taken together are equal to two right ones, and therefore make 180 *Degrees.*

Fig. 52. Draw forth one Side A B unto F. The external Angle F B C is equal [b] to the two internal oppoſite ones, A and C. But F B C with A B C make [c] Angles equal to two right ones. Therefore the two A and C with the ſame C B A make Angles equal to two right ones. *Q. E. D.*

[b] By the firſt Part of this.

[c] *Per* 13. *l.* 1.

Fig. 53. Or thus: Draw the Line H M parallel to A C, the alternate Angles as well O and A, as N and C [d] are equal. But O, Q, N make Angles [e] equal to two right ones. Therefore alſo A, C, Q are equal to two right ones. *Q. E. D.*

[d] *Per* 27. *l.* 1.

[e] *Corol.* 1. *Prop.* 13. *l.* 1.

Corollaries.

4. THE three Angles of any one Triangle taken together are equal to the three Angles of any other Triangle taken together.

5. If in a Triangle one Angle be right (or obtuſe) the reſt are acute.

6. If in a Triangle one Angle be right, the two other Angles together make one right Angle.

7. In every Triangle, the Angle, which is right, is equal to the other two taken together.

8. When you know of how many Degrees one Angle of a Triangle is, you know at the ſame time how many Degrees the two other Angles, as taken together, do make up. And ſo on the contrary, when you know how many Degrees two Angles of a Triangle taken together do make up, or what is the Sum of them, you know at the ſame time of how many Degrees the third Angle is.

9. When two Angles of one Triangle either ſeverally or together are equal to two Angles of another Triangle, the third Angle of one Triangle is alſo equal to the third of the other.

10. When two Triangles have one equal Angle, the Sums alſo of the reſt of the Angles are equal.

11. When in an *Iſoſceles* the Angle contained by the equal Sides is a right one, the two others are each of them half-right Angles. And the Angles of an *Iſoſceles*, which are at the Baſe, are always acute.

12. In an equilateral Triangle, each Angle is two thirds of a right Angle. For it is one third of two right ones; therefore it is two thirds of one right one.

13. Hence a right Angle (BAC) is eaſily divided into three equal Parts; if upon AC be made the equilateral Triangle Z; for, ſeeing FAC is two thirds of one right one, BAF ſhall be one third of a right one. Fig. 54.

14. The Perpendicular AF is the ſhorteſt of all Lines, which can be drawn from the Point (A) unto ſome right Line. For ſeeing the Angle F is a right one, AOF ſhall by Corollary Fig. 55.

[f] *Per* 19. *l.* 1.

lary the 5th be an acute one. Therefore [f] AF is ſhorter than any other, as AO.

15. Only one Perpendicular can fall from one Point unto one right Line. This is manifeſt out of the foregoing Corollary.

Fig. 86.

16. *Hence alſo we learn to determine the Parallax of the Stars, or the Difference of their true and apparent Place. Let A be the Centre of the Earth, B the Place of the Obſerver, upon the Surface of the ſame. Let DBC be the Angle of the Star C according to Obſervation, or the* viſible *angular Diſtance thereof from the vertical Point; when in the mean while DAC is the* true *angular Diſtance. Now the external Angle DBC, which is given from Obſervation, is equal to the Angles BAC and BCA taken together; and conſequently the Angle BCA is the Difference of the Angles DBC and DAC. If therefore we ſhall from aſtronomical Tables ſeek the Angle DAC, or what, at the time of Obſervation, is the true angular Diſtance of the Star from the vertical Point, when the Angle DBC is at the ſame time known by means of the Quadrant, the Difference of thoſe Angles BCA, which we call the* Parallax, *will likewiſe be known.* Q. E. I.

Scholium.

BY the Teſtimony of *Eudemus* an ancient Geometrician, *Pythagoras* was the Finder-out of this Propoſition, which indeed is a Theorem moſt excellent in itſelf, moſt fruitful in its Conſectaries, and of uſe in all Parts of the Mathematicks. *Ariſtotle* very frequently makes mention of it, who alſo puts it for an Example of the moſt perfect Demonſtration. But like as from this Propoſition we have already learned, how many right Angles the Angles of a Triangle are equivalent to; ſo, by the Help of the ſame, it will in the three following Propoſitions be manifeſt, how many right Angles the Angles of any rectilinear Figure whatſoever, whether internal or external, do make.

Theorem 1.

IN every quadrangular Figure the four Angles together make four right ones. Fig. 56.

For if through the oppoſite Angles you draw the right Line BF, this will cut the Triangle into four Quadrangles, without forming any new Angles, whoſe Angles together do[g] make four right Angles.

[g] *Per* 32. *l.* 1.

Theorem 2.

ALL the Angles together of every right-lin'd Figure make twice ſo many right ones, abating four, as are the Sides of the Figure.

From any Point A within the Figure let there be drawn unto the Angles of the Figure right Lines, which ſhall cut the Figure into ſo many Triangles as it hath Sides, and make no more Angles but thoſe of the Centre. Wherefore when each of the Triangles contains two right Angles[h], they muſt all together contain twice ſo many right Angles as there are Sides. Now the Angles about the Point A[i] do make four right Angles. Therefore, if from the Angles of all the Triangles you take away the new Angles which are about A, the remaining Angles, which indeed do alone conſtitute the Angles of the Figure, will make twice ſo many Fig. 57.

[h] *Per* 32. *l.* 1.

[i] *Corol.* 3. *Prop.* 13. *l.* 1.

right Angles, excepting ſour, as are the Sides of the Figure.

Hence it appears, that all right-lin'd Figures of the ſame Species, or Number of Sides and Angles, have the Sums of their Angles equal. Which thing is worthy of Admiration.

The Practice is thus: Double the Denominator of the Figure, and from the Product take away four; the Remainder is the Number of the right Angles, which the internal Angles of the Figure do make.

Theorem 3.

Fig. 58. ALL the external Angles of any right-lin'd Figure whatſoever taken together do make up four right Angles.

For each of the internal Angles of the Figure does[k], with its reſpective external one, make two right Angles. Therefore all the internal ones, together with all the external ones, do make up twice ſo many right Angles as are the Sides of the Figure. Now, by the Precedent, the internal ones, together with four right Angles added to them, make twice ſo many right Angles as are the Sides of the Figure. Therefore the external Angles are equal to four right ones.

[k] *Per* 13. *l.* 1.

Wonderful truly is this Property of right-lin'd Figures; from whence it follows alſo, that all the right-lin'd Figures of any Species whatſoever have the Sums of their external Angles equal. And therefore the three external Angles of a Triangle are equal to the thouſand external Angles of a thouſand-ſided Figure. Which Obſervation is altogether worthy of Admiration.

PROP.

PROP. XXXIII. Theorem.

IF two right Lines, which are equal and parallel, as (AB, CF) be joined by two others (AC, BF); theſe alſo will be equal and parallel. Fig. 59.

Let AF cut the Parallels AB, CF. In the Triangles Q, R, the alternate Angles BAF, CFA [l] will be equal. Now the Side AB is ſuppoſed equal to the Side CF, and AF is common to both Triangles. Therefore [m] the Baſes BF, AC are equal. (Which is the firſt Part.) And then the Angles at the Baſes AFB, FAC are equal; and therefore AF falling upon the right Lines AC, and BF, makes the alternate Angles AFB, FAC equal. Therefore AC, BF are alſo [n] parallel. Which is the other Part.

[l] *Per* 27. *l.* 1.

[m] *Per* 4. *l.* 1.

[n] *Per* 28. *l.* 1.

Coroll. (1.) *Hence we learn to meaſure as well the Heights of Mountains above the Horizon as their horizontal Lines. Let ABC be the Side of a Mountain, to which apply a great Square, or ſome Inſtrument equivalent thereto ADB. Then ſhall AD be equal to HB, and DB equal to AH. Then coming unto the lower Part, which is from the Point B unto the Point C, practiſe as before. So ſhall EB be equal to CF, and EC be equal to BF. Which done, the Sides parallel to the Horizon, AD, BE,* &c. *added together will give the horizontal Line GC; and the perpendicular Sides BD, EC,* &c. *added together will give the Height AG.* Fig. 87.

Coroll. (2.) *Hence alſo we learn to eſtimate the Compoſition of Motions. Let a Body placed at A be* Fig. 59.

be driven in the ſame Moment of Time by the Force AC, according to the Direction of the Line AC, and by the Force AB according to the Direction of the Line AB. From the Conjunction of theſe two Forces it will deſcribe the Diagonal AF. For in this Line of its Motion neither of the Forces is changed. For the Body at F is equally diſtant from both the Lines of Direction AC, AB, as if it had been driven by either of the Forces ſeparately; which thing can be ſaid of no other Point. And this Corollary doth ſo fully agree with aſtronomical and other mechanical Phænomena, that it is worthily reckoned by the famous Sir Iſaac Newton, *as a Foundation of his Geometrical Philoſophy.*

PROP. XXXIV. Theorem.

Fig. 59.

IN every Parallelogram the oppoſite Sides and Angles are equal, and it is cut into two equal Parts by the Diameter.

Becauſe AB, CF are [o] parallel, and AF falls upon them, the alternate Angles BAF, CFA are [p] equal. Likewiſe becauſe AC, BF [q] are parallel, and upon them falls the Line AF, the Alternates CAF, BFA [r] are parallel. Therefore the whole Angle BAC is equal to the whole Angle BFC. In the ſame manner B and C are ſhew'd to be equal. Which was the firſt Part.

[o] *Per Def.* 35.
[p] *Per* 27. *l.* 1.
[q] *Per Def.* 35.
[r] *Per* 27. *l.* 1.

Now becauſe it hath been already ſhew'd that the Triangles Q R, which have one common Side AF, have alſo the Angles adjacent to the common Side equal, BAF to CFA; and CAF, to BFA; the Sides likewiſe ſhall be equal, A B to F C, and B F to A C; and thus the whole Triangles are equal. Which was the ſecond Part.

Scholium.

PARTLY from this Theorem, and partly from a Definition to be premis'd to the ſecond Book, the Meaſuring of a right-angled Parallelogram is eaſily deduced. The Area thereof being produced by the Multiplication of the two contiguous Sides A F, A C one by another. *E. G.* Let A F be a Line of 8, A C a Line of 4 Feet. Multiply 8 by 4, there ariſes 32 ſquare Feet for the Area of the Rectangle. Fig. 60.

But the Area of a Square is had from the Multiplication of the Side F I by itſelf; as if F I be of 5 Feet, multiply 5 into itſelf, there will ariſe 25 ſquare Feet for the Area of the Square. Fig. 61.

The Demonſtration is manifeſt from this Propoſition, if parallel Lines be drawn through the Diviſions of the Sides.

Corollary. *Hence Surveyors do eaſily divide the Area of a Field when it is a Parallelogram. For let A B C D be the parallelogram Field: A D the Diameter or diagonal Line of the ſame, the middle Point whereof is marked F. Whatſoever right Line, as E G, paſſeth through the Point F, it divides the Field into equal Parts E A C G, E B D G. For the Triangle A B D is equal to the Triangle A C D, and* [s] *the Triangle A E F equal to the Triangle G F D. If therefore to the Trapezium E B D F, inſtead of the Triangle A E F, you ſhall add the Triangle which is equal to* Fig. 88.

[s] *Per* 26. *l.* 1.

to it GFD, you will not change the Area; but the Trapezium EBDG will be equal to the Triangle ABD or to half the Parallelogram, and conſequently equal to the Trapezium AEGC. Q. E. I.

PROP. XXXV, XXXVI. Theorems.

Fig. 62.

Parallelograms upon the ſame or equal Baſes (AB) and between the ſame Parallels (CQ, AX) are equal.

Becauſe AL, BQ[t] are parallel, and CQ cuts them, the external Angle CLA ſhall[u] be equal to the internal one FQB. Then, becauſe as well CF as LQ are equal[x] to the ſame AB, CF is equal to LQ. Add then FL to both, the whole Lines CL, FQ are equal. Moreover, AL, BQ are equal[y]. Therefore the Triangles CLA, FQB[z] are equal. Therefore taking away the common Triangle FOL, the Planes FOAC, QBOL remain equal: To each of which Trapeziums add the Triangle AOB, the whole Parallelograms ACFB, ALQB become equal. *Q. E. D.*

This Propoſition will be made univerſal, *Prop.* 1. *l.* 6. Beginners may here obſerve, that although of two Parallelograms which are between the ſame Parallels infinitely produced, and upon the ſame Baſe, one of them be extended unto an infinite Length, it ſtill remains but equal to the other, by the Force of the preſent Demonſtration.

[*From hence it follows, that two Cities, in Magnitude equal, may ſo much differ in Compaſs, that the*

[t] *Per Def.* 35.

[u] *Per* 27. *l.* 1.

[x] *Per* 34. *l.* 1.

[y] *Per* 34. *l.* 1.

[z] *Per* 4. *l.* 1.

the Circumference of one may exceed that of the other an hundred or a thouſand Times. If, for Inſtance, one be of a ſquare Figure or rectangular; but the other a Parallelogram, betwixt the ſame Parallels indeed with the former, but very oblong.

Moreover, it hence follows, that Figures of equal Compaſs round may contain Area's vaſtly different.

Scholium.

FROM this Theorem we may learn to meaſure any Parallelogram. For the Area of it is produced from the perpendicular Altitude Q X, or C A multiplied into Baſe A B. Fig. 62.

For the Area of the Rectangle C B, which is equal to that of the Parallelogram B L, is made [a] by A C, multiplying A B. Therefore, &c.

[a] By the foregoing Scholium.

PROP. XXXVII, XXXVIII. Theorems.

TRIANGLES (ACB, AFB) upon the ſame or equal Baſes (AB), and between the ſame Parallels (CI, AZ) are equal. Fig. 63.

Draw the Lines B L, B I parallel to the Sides A C, A F. The Parallelograms A C L B, A F I B [b] are equal. But the given Triangles are Halves of thoſe Parallelograms [c]. Therefore the given Triangles [d] are equal.

This Propoſition will be made univerſal, *Prop.* 1. *l.* 1. Let Beginners mark the ſame Thing here concerning Triangles, which we bid them to note in the foregoing Propoſition concerning Parallelograms.

[b] By the foregoing.

[c] *Per* 34. *l.* 1.

[d] *Per Axiom* 6.

Fig. 89. Corollary (1.) *Hence Surveyors eaſily divide the Area of a triangular Field. Let A B C be the Field, and let the Baſe B C be biſected in D. The Triangles ABD, ADC upon the equal Baſes B D and D C, and having a common Top A, or being between the ſame Parallels, are equal.* Q. E. F.

Fig. 90. Coroll. (2.) *Hence we alſo gather, with the famous Sir* Iſaac Newton, *that the Area's,* ~~*with*~~ *all Bodies whatſoever, that revolve round about an immoveable Centre, towards which they are impelled, do deſcribe, are both in immoveable Planes, and are proportional to the Times of Deſcription. For let the Time be divided into equal Parts, and, in the firſt equal Part of Time, let the Body by the impreſſed Force deſcribe the right Line A B. The ſame Body in the ſecond Part of Time, if nothing hindered, would go forwards ſtrait unto c, deſcribing the Line B c equal to A B; ſo that the Area's made by Lines drawn from the Centre A S B, B S c*[e] *would be equal. But when the Body comes unto B, let the Force act with one ſingle Impulſe, but a great one, and make the Body to deflect from B c, and to go forwards in the right Line B C;* i. e. *let the centripetal Force be in that Place to the Force before impulſed, as C c or B g is to B c; in this Caſe the Body will*[f] *deſcribe the Diagonal BC. Let there be drawn parallel to B S the right Line C c meeting B C in C. In the ſecond Part of Time compleated, the Body will be found in the Point C, in the ſame Plane with the firſt Triangle S A B. Join S C. The Area made by a Ray drawn from the Centre; that is, the Triangle S B C will be equal to*[g] *S B c, and conſequently to the firſt Angle S A B*[h]. *By the ſame Argument the Body in the third equal Part of Time would by its preſent Force reach from*

ck w =

[e] *Per* 37. *l.* 1.

[f] *Per Corol.* 2. *Prop.* 33. *l.* 1.

[g] *Per* 37. *l.* 1.

[h] *Per Axiom* 1.

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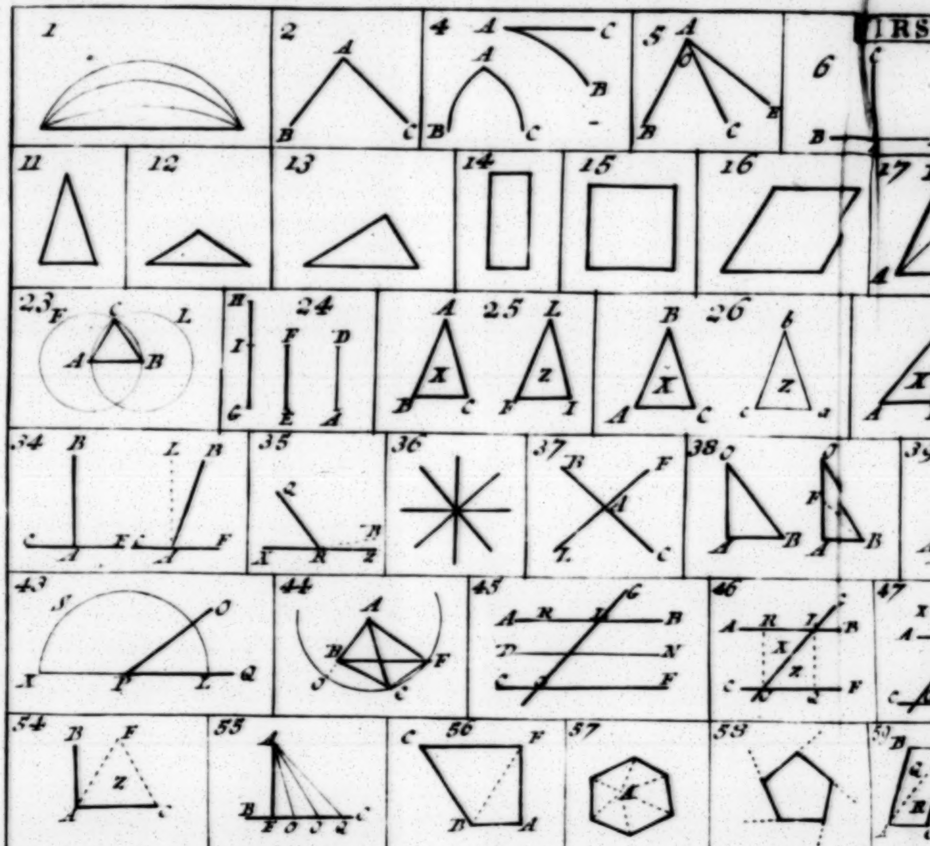
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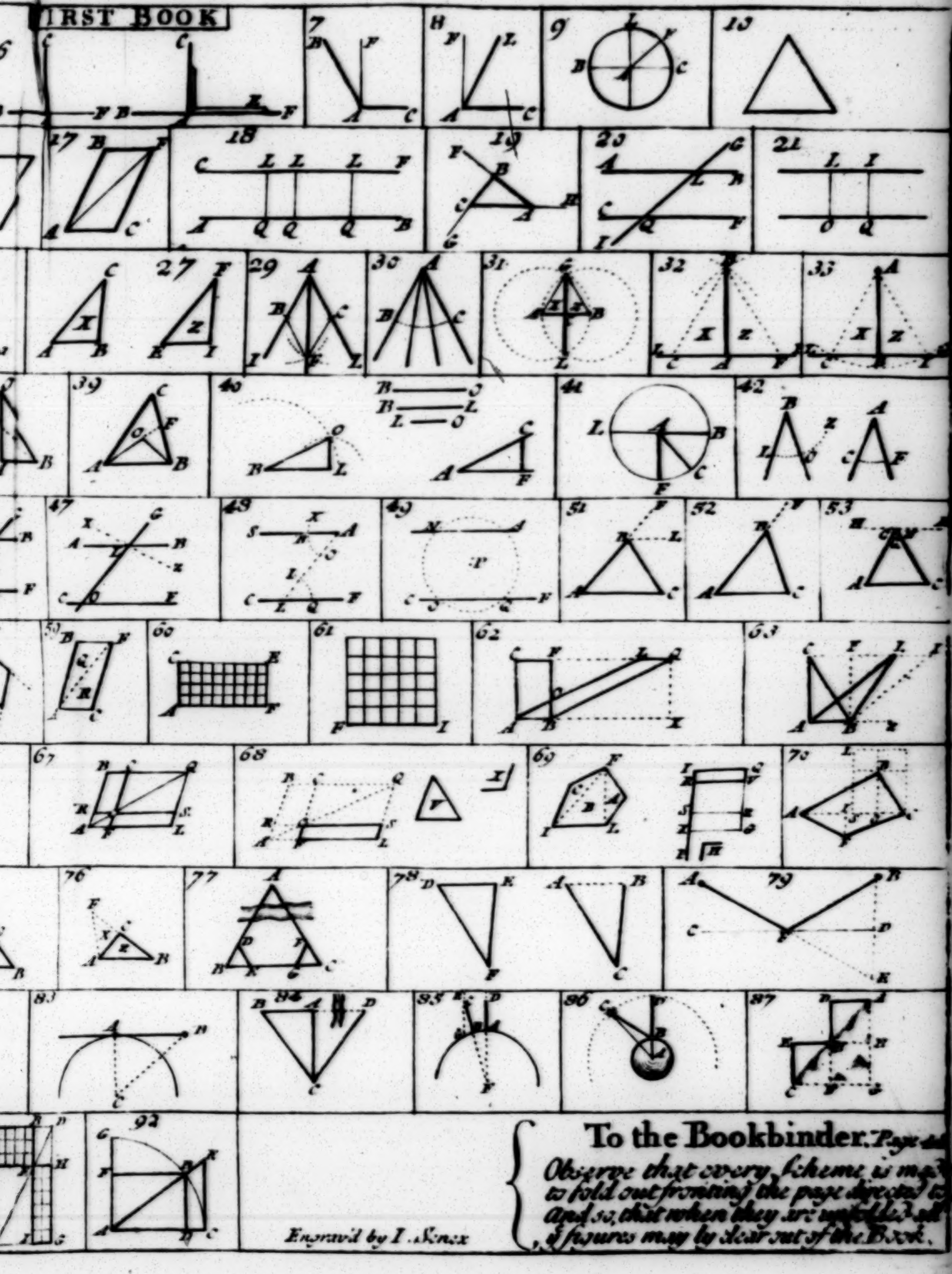

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To the Bookbinder. *Page*

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from C unto d; ſo that the Line C d ſhould be equal to the Line B c or A B. But if the centripetal Force, whether it be greater or leſſer, does again act upon it in the Point C, in the End of the third Part of Time it will be found ſomewhere in the Line D d, parallel to S C; and therefore as before, ſuppoſing the ſaid Force to be equal or unequal to what it was before, it will be found to have deſcribed the Diagonal C D, and will be found in the Point D, and a Ray being drawn from the Centre, the Triangle S D C will be equal to that S d C; and conſequently to the others S C B, S A B which are equal one to the other. In like manner, if the centripetal Force act ſucceſſively in the Points D, E, F, and be the Cauſe that the Body in the ſeveral Parts of Time reſpectively deſcribes the Diagonals D E, E F, &c. *the Area's now made as afore will be in the ſame Plane, and Triangles will be deſcribed equal to the former Triangles. Therefore in equal Times equal Area's are deſcribed in an immoveable Plane; and ſo the Sums of the Area's S A D S, S A F S will be amongſt themſelves as the Times wherein they were deſcribed. Now let the Number of the Triangles be increaſed, and their Wideneſs diminiſhed infinitely; both that laſt Perimeter of them A B C D E F will be a curve Line, and the Area's deſcribed in one and the ſame immoveable Plane will in this Caſe alſo be proportional to the Times as well as before.* Q. E. D.

PROP. XXXIX, XL. Theorems.

Fig. 64.

EQUAL Triangles (A C B, A F B) upon the ſame or an equal Baſe (A B) and on the ſame Side, are between the ſame Parallels, (A B, C F).

If you deny it, let C L be parallel to A B and let B L be drawn. Then A L B is equal to A C B[i]. But by the Hypotheſis A F B is equal to A C B. Therefore A L B and A F B are equal; *i. e.* a Part is equal to the Whole. Which cannot be. Therefore, &c.

[i] By the foregoing.

[Coroll. (1.) *Hence alſo, with the famous Sir* Iſaac Newton, *we gather, that all Bodies which are moved in curve Lines, and deſcribe Area's about ſome Centre proportional to the Times, are perpetually urged and preſs'd by a Force impelling towards the Centre. For becauſe of the Equality of the Triangles S C B, S c B deſcribed upon the ſame Baſe S B, the Points C and c ſhall be in a Line C c which is parallel to the Baſe; and ſo the Figure B c C g ſhall be a Parallelogram; the Sides whereof B c and B g are*[k] *the Lines of the Directions of the Forces, and B C is the Diagonal. The Body therefore is urged unto C by the Force B g which tends unto S the Centre. And ſo in all the Points, C, D, E, F.* Q. E. D.

[k] *Per Coroll.* 2. *Prop.* 33. *l.* 1.

Coroll. (2.) *Seeing therefore, in the Motion of the primary Planets, the Area's made by Rays, or right Lines drawn from them unto the Sun, are always proportional to the Times, as all Aſtronomers know; the Planets are urged by a perpetual Force, which tends to the Sun. And the ſame thing is equally true of the ſecondary Planets with reſpect to their primary ones.*]

PROP. XLI. Theorem.

Fig. 65.

If a Triangle (A F B) be in the ſame Parallels with a Parallelogram (A L), and have the ſame or an equal Baſe (A B), it is half of the Parallelogram.

Draw C B. The Triangles A F B, A C B are[l] equal. But A C B is half of the Parallelogram A L[m]. Therefore A F B alſo is half of A L. *Q. E. D.*

[l] *Per* 37, 38. *l.* 1.
[m] *Per* 34. *l.* 1.

Scholium.

Fig. 65. FROM this Propoſition, with the Scholium of *Prop.* 35, we learn, that the Area of whatſoever Triangle, as A F B, is produced from half the Altitude F I multiplied into the Baſe A B, or half the Baſe multiplied into the Altitude. Wherefore one Side of a Triangle being known, and the Height, that is, the Perpendicular which falls upon the known Side from the oppoſite Angle, the Meaſure of the Triangle is given. As if the Baſe A B be of an 100 Feet, the Height F I 85. multiply half the Baſe 50 by 85, and you have the Area of the Triangle A F B = 4250 Feet ſquare. Further, the Altitude of a Triangle, when the Area of it is in all Points acceſſible, may be known mechanically as well as the Sides. But, if the Area of it cannot be gone over, the Height may be found geometrically by 12 and 13. *Lib.* 2. as we ſhall there ſhew.

In a rectangle Triangle, the Height is the ſame with either of the Sides about the right Angle. Half of this therefore multiplied into the other Side adjacent to the right Angle will give the Area of the Triangle.

PROP.

PROP. XLII. Theorem.

Fig 66. *TO make a Parallelogram with an Angle equal to a given one* (O); *and equal to a given Triangle* (*A C B*).

Biſect the Baſe A B in F. Through C draw CX parallel [n] to AB. Make the Angle B A L equal to the given one O [o]. Draw F I parallel [p] to A L. A L I F ſhall be that which was ſought for.

For let F C be drawn. The Parallelogram A I hath an Angle L A F equal to the given one O, and is equal to the given Triangle A C B; ſeeing that as well as the Triangle A C B [q] as the Parallelogram A I [r] is double to the ſame Triangle A C F.

[n] *Per* 31. *l.* 1.
[o] *Per* 23. *l.* 1.
[p] *Per* 31. *l.* 1.
[q] *Per* 38. *l.* 1.
[r] By the foregoing.

Corollary.

Fig. 66. THE Triangle A C B being given, a Rectangle equal to it is had, if there be drawn a Line parallel to the Side A B, and A B being biſected in F, the Perpendicular B Q be erected. For the Rectangle under F B and Q B will be equal to the Triangle A C B.

PROP. XLIII. Problem.

Fig. 67. *IN a Parallelogram* (*as B L*) *the Complements* (*B O, O L*) *of thoſe Parallelograms which are about the Diameter* (*R F, C S*) *are equal.*

If through any Point of the Diameter A Q, as the Point O, C F be drawn parallel to the Side A B, and R S parallel to the Side B Q; the whole Parallelogram B L is divided into four Parallelograms, whereof two are about the Diameter RF, CS, the other two B O, O L are the Complements of theſe unto the whole Parallelogram B L.

Their Equality is thus proved: The Triangles [s] A B Q, A L Q are equal. Likewiſe the Triangles A R O, O C Q [t] are equal to the Triangles A F O, OSQ. Therefore if from the Equals A B Q, A L Q you take away Equals, on this Side A R O, O C Q, on that A F O, O S Q; then B O and O L ſhall remain equal. *Q. E. D.*

[s] *Per* 34. *l.* 1.

[t] By the ſame.

PROP. XLIV. Problem.

UPON *a given right Line* (*O S*) *to conſtitute a Parallelogram, in a given Angle* (*X*), *which Parallelogram ſhall be equal to a given Triangle* (*V*).

Fig. 64.

Make a Parallelogram [u] R C equal to the given V, having its Angle R O C equal to the given one X, and join the Side R O directly to the given Line OS, ſo as to make one right Line therewith. Then through S draw S Q [w] parallel to O C, which S Q let B C meet when it is produced unto Q. Then let a right Line drawn through Q and O meet BR produced unto A. Which done, through A draw A L parallel to OS, which A L let CO and QS

[u] *Per* 42. *l.* 1.

[w] *Per* 31. *l.* 1.

meet when it is produced unto F and L; the Parallelogram OL is that which is required.

[x] By the foregoing. For OL [x] is equal to RC, that is, by the Conſtruction, to the given Triangle V, and is
[y] *Per* 15. *l.* 1. at the given Line OS; and [y] the Angle FOS is equal to the Angle ROC; that is, by the Conſtruction, equal to the given Angle X.

Scholium. *This Propoſition contains a certain Geometrical Diviſion. For, in the vulgar Arithmetical Diviſion, the Number to be divided may juſtly be conſidered as being a certain Rectangle.*
F: 91- E. G. *Let the Rectangle AB, comprehending* 12
ſquare Feet, be to be divided by 2; i. e. *a Rect-*
Fig. 91. *angle is to be found equal to that AB of* 12 *ſquare Feet, one of whoſe Sides ſhall be only* 2 *Feet: From whence it comes to be enquired of what Number the Side ſought ſhall conſiſt; which Side is to be eſteemed a certain Quotient of this Diviſion. Which Thing is performed Geometrically after this manner: With a Pair of Compaſſes take the Line BD of two Feet, and draw the Diagonal DEF. The Line AF is that which is ſought*
[z] *Per* 43. *l.* 1. *for. For the Complements EG and EC are* [z] *equal; and in the Rectangle EG one Side EH is equal to the Line BD which is of* 2 *Feet; and the Side EI is equal to AF.*

This Kind of Diviſion is called Application, *becauſe the rectangular Space AB is* applied *to the Line BD or EH; and hence it comes that Diviſion is often named* Application; *Reſpect being had to the Practice of the old Geometricians, who always made more Uſe of Geometrical Conſtruction, which requires only a Rule, and a Pair of Compaſſes, than of Arithmetical Computation, which is performed by Numbers.*

PROP.

PROP. XLV. Problem.

UPON a given Line (IQ) and in a given Angle (H) to make a Parallelogram equal to a given Rectilineal Figure (CBA). Fig. 69.

Reſolve the given Rectilinear into the Triangles A, B, C, by drawing the right Lines FL, FI.

Upon the given Line IQ in the given Angle H make [a] the Parallelogram IV equal to the Triangle A. Then, the right Line IR being produced infinitely towards P, upon the right Line RV in the Angle VRP [b] make the Parallelogram RZ equal to the Triangle B. Again, upon the Line SZ with the Angle *ZS*P make the Parallelogram SG equal to the Triangle C. This done, I ſay IG is the Parallelogram ſought for.

[a] *Per* 44. *l.* 1.
[b] By the ſame.

For [c] the Angle ZVR is equal to its alternate IRV. But [d] QVR and IRV are equal to two right Angles. Therefore alſo QVR and ZVR are equal to two right ones. Therefore [e] QV and ZV fall directly ſo as to make one right Line. After the ſame manner I might ſhew that QZ and ZG make one right Line. Therefore the whole QVZG is one right Line, and is alſo parallel to IX, ſeeing by the Conſtruction QV is parallel to IP. Now XG alſo [f] is parallel to IQ. Seeing XG is parallel to SZ, and SZ to RV, and RV to IQ.

[c] *Per* 27. *l.* 1.
[d] By the ſame.
[e] *Per* 14. *l.* 1.
[f] *Per.* 30. *l.* 1.

IG therefore [g] is a Parallelogram; but that it is ſuch an one as was required, is manifeſt from the Conſtruction.

[g] *Per def.* 35.

[Coroll.

[Coroll. *Hence is eaſily found the Exceſs whereby a greater rectilinear Figure exceeds a leſſer. To wit, if unto the ſame right Line IQ be applied Parallelograms reſpectively equal to the two right lin'd Figures. For that Parallelogram by which the greater Rectilinear exceeds the leſſer will give the Difference of them.* Q. E. I.]

Scholium.

WE will here add a Problem that will be uſeful for the Practice of Propoſition 14. l. 2.

Fig. 70. A quadrangular Figure BF being given, to deſcribe an equal Rectangle.

Reſolve it into Triangles by the right Line AC. From the oppoſite Angles let down the Perpendiculars BO, FI. Biſect AC in S. From S erect the Perpendicular SL, equal to the two BO, FI put together. The Rectangle comprehended under LS and SA is equal to the given BF. The Demonſtration appears out of Propoſition 41.

PROP. XLVI. Problem.

Fig. 71. *FROM a given right Line (AB) to deſcribe a Square.*

Erect two Perpendiculars equal to the given AB; to wit, AC, BE, then join CE. I ſay the Thing is done.

For, ſeeing the two Angles A and B are[h] right ones, AC and BE ſhall[i] be parallel; but they are alſo[k] equal. Therefore CE and AB are[l] parallel and equal. Therefore the Figure

[h] By the Conſtruction.

[i] *Per* 29. *l.* 1.

[k] By the Conſtruction.

[l] *Per* 33. *l.* 1.

gure is a Parallelogram and Equilateral. But all the Angles alſo are right ones (for, ſeeing A and B are right Angles, the oppoſite ones[m] E and C are right alſo). Therefore the Figure A E is a Square.

[*In the ſame manner you may eaſily deſcribe a Rectangle which hath the two unequal Sides given.*]

PROP. XLVII. Theorem.

Fig. 72.

IN every right-angled Triangle (as ABC) the Square of the Side (AC) which is oppoſite to the right Angle is equal to the two Squares together of the two other Sides (AB, CB.)

Let I C and B F be drawn; and B E parallel to A F. Now if to the right and therefore equal Angles I A B, F A C, there be added the common Angle B A C, the Wholes I A C and F A B ſhall be equal. But in the Triangles I A C, F A B, the Sides which contain thoſe equal Angles are equal[n] amongſt themſelves, to wit, I A, C A, to B A, A F, each to each. Therefore the Triangles I A C, F A B[o] are equal. Which becauſe they ſtand upon the ſame Baſes I A, F A with the Parallelograms A B L I and Z A F E, and between the ſame Parallels I A, L B C, and A F, E Z B, they are Halves[p] of thoſe Parallelograms. Therefore the Parallelograms A B L I, Z A F E, as being Doubles of Equals, are equal betwixt themſelves. By the ſame Reaſoning the right Lines A X, B R being drawn, it may be ſhewn that the Parallelograms E C, B X are equal.

[m] *Per* 34. *l.* 1.

[n] *Per def. Square.*

[o] *Per* 4. *l.* 1.

[p] *Per* 41 *l.* 1.

Therefore the whole AR is equal to IB and BX together. *Q. E. D.*

It was taken for granted that LBC is parallel to IA, in order to which LB and BC muſt be one right Line. Now that they are ſo is manifeſt from the 14th, ſeeing the Angles LBA and CBA are both right ones by the Hypotheſis.

Scholium.

THIS Theorem (which *Prop.* 31. *l.* 6. *Euclid* extends unto all like or ſimilar Figures) is commonly call'd the *Pythagoric* Theorem, from *Pythagoras* the Inventor of it; who, as is atteſted by *Proclus*, *Vitruvius*, and others, offer'd Sacrifices to the Muſes, as ſuppoſing himſelf to have been helped by them in ſo excellent an Invention; in which thing he ſhew'd himſelf to be ignorant of God, the Lord of Sciences, the true and only Author of all Wiſdom; or certainly if he knew him, he glorified him not as God. There is frequent and notable Uſe of this Theorem through all the Mathematicks; and in particular it opens a Way unto the Knowledge of incommenſurable Magnitude, a main Secret of Geometrical Philoſophy.

That the Side of a Square incommenſurable to the Diameter is a thing much celebrated amongſt the old Philoſophers, *Ariſtotle* and *Plato* eſpecially; inſomuch that *Plato* would ſay, that he who knows not this is not a Man, but a Beaſt. Now the Knowledge of this Myſtery ſeems to have taken its Riſe out of this 47th Propoſition. For ſeeing in the Square
Fig. 71. AE the Angle A is a right Angle, the Square of the Diameter CB ſhall be equal to both the

Squares of the Sides, A B, A C, and therefore double to one of them. Wherefore ſeeing the Square of C B is 2, and the Square of the Side A B is 1 or Unity, the Diameter C B ſhall be the ſquare Root of 2, and the Side A B the ſquare Root of Unity, or Unity itſelf; the *Ratio* of which Quantities (as it will be demonſtrated in its Place) cannot be explicated in Numbers, and therefore they are incommenſurable.

And by this one Argument alone, if all others were wanting, it might evidently be made out, that Geometrical Magnitudes cannot be made up of a definite Number of Points; for otherwiſe none would be incommenſurable; foraſmuch as a Point would be the common Meaſure of all.

To theſe Things we will ſubjoin three Problems which are deduced out of the preſent Propoſition, and are of frequent Uſe.

Problem 1.

IF any Number of Squares are given, to make one equal to them all together. Fig. 73.

Let there be three or more Squares given, whoſe Sides are A B, B C, C E. Make the right Angle F B Z having indefinite Sides, and unto the Sides of it transfer A B and B C, and then join A C. The Square of A C ſhall be equal to the Squares of A B and B C together [q]. Then transfer A C from B unto X, and C E the third given Side transfer from B unto E, and join E X; the Square of E X ſhall be equal to the Squares of E B (or E C) and B X together; that is, equal to the three given

[q] *Per* 47. *l.* 1.

Squares, whoſe Sides are A B, B C, C E: And ſo on as long as you pleaſe.

Problem 2.

Fig. 74. TWO unequal right Lines being given (A B, B C) to determine that Square, whereby the Square of the greater (A B) exceeds the Square of the leſs (B C).

From the Centre B with the Interval A B deſcribe a Circle. Then from C erect a Perpendicular C E, cutting the Circumference in E. The Square of C E is the Exceſs or Difference which is ſought for.

For let B E be drawn. The Square of B E, that is, of A B, is equal to the Squares [r] of B C and C E together. Therefore, *&c.*

[r] *Per* 47. *l.* 1.

Problem 3.

Fig. 75. ANY two Sides of a right-angled Triangle being known, to find the third.

Let the Sides containing the right Angle be A B, A C, the one of 6 Feet, the other of 8. You are to find of how many Feet the Side C B, which is oppoſite to the right Angle, is. To do which, multiply 6 and 8 each of them by itſelf. From which Multiplication there will ariſe for the Squares of thoſe two Sides 36 and 64; the Sum of which is an 100. The ſquare Root of an 100, which is 10, gives the Feet of the Side B C, whoſe Quantity was ſought. This Demonſtration offers itſelf in and from this 47th Propoſition; for the Sum of the Squares B A and C A is equal to the Square of B C. Therefore the Root of the Sum of them is equal to the Root or Side B C.

Then let the Sides AB, BC be known, the one of 6 Feet, the other of 10, you are now to find AC. Take the Square of the Side AB which is 36, out of the Square of the Side BC = 100. The Remainder 64 ſhall be the Square of the Side AC. The Root therefore of 64, which is 8, gives the Feet of the Side AC.

Corollary, *From hence we derive the Original of the Tables of Sines, Tangents, and Secants. For, for Inſtance, let AC the Semidiameter of the Circle be of* 100,000 *Parts, and the Angle BAD of* 30 *Degrees. Becauſe the Chord or Subtenſe of* 60 *Degrees*[s] *is equal to AC the Semidiameter; BD the Sine of* 30 *Degrees ſhall be equal to half the Semidiameter, or* ½ *AC; and therefore ſhall contain* 50,000 *Parts. But now, in the right-angled Triangle ADB, the Square of AB is equal to the Squares of AD and BD. Wherefore let the Semidiameter AB be ſquared (by multiplying* 100,000 *by* 100,000) *and from that Square ſubtract the Square of BD. The Remainder ſhall be the Square of AD, or of the Coſine equal to it BF; out of which extract the ſquare Root, and you will have the Line BF or AD. Then by this following Analogy, AD : BD :: AE : CE, or AD : BD :: AC : CE, will be had the Tangent CE. And then laſtly, if the Square of AC be added to the Square of CE, the Root of the Sum being extracted will be the Secant AE.* Q. E. I.

Fig. 92.

[s] *Per Corol.* 1. *Prop.* 15. *l.* 4. *and Corol.* 2. *Prop.* 3. *l.* 3.

PROP. XLVIII. Theorem.

Fig. 76.

IF *in a Triangle the Square of one of the Sides (AB) be equal to the two Squares of* *of the other Sides (AC, BC) taken together, the Angle (ACB) which the two other Sides contain, is a right Angle.*

If not, the Angle ACB will be greater or leſſer than a right Angle. In either of which Caſes (as it will be demonſtrated, *Prop.* 12, 13, *l.* 2. which Propoſitions depend not on this) the Square of AB will not be equal to the Squares of AC, BC together; which is contrary to the Hypotheſis.

Or thus: Draw FC perpendicular to AC, and equal to CB, and join AF. The Square of AF is [t] equal to the Squares of FC, CA together; that is, [u] to the Squares of BC, CA; that is, by the Hypotheſis, to the Square of AB. Therefore the right Lines AF, AB are equal. Becauſe therefore the Triangles are mutually equilateral, the Angles at C [w] are equal. Therefore they are both right Angles [x]. *Q. E. D.*

[t] *Per* 47. *l.* 1.

[u] By the Conſtruction.

[w] *Per* 8. *l.* 1.

[x] *Per Def.* 14.

THE

THE ELEMENTS OF EUCLID.

BOOK II.

THIS Book is ſmall in Bulk, but great indeed in the Excellence and Uſefulneſs of its Theorems. Young Beginners will not, I know what I ſay, be at firſt able to diſcover it; but, being further advanced, they will from their own Experience, and with the greateſt Certainty, apprehend that it is moſt true.

A DEFINITION.

Fig. 60. l. 1.

A Right-angled Parallelogram (as A E) (which is wont ſimply, and without any Addition, to be call'd a Rectangle) is ſaid to be contain'd under the two Lines A C, A F) which determine the Magnitude of it:

For the one of them A C determines the Height, the other A F the Breadth of it. Now if the Side A C be underſtood to be carried perpendicularly along the whole A F, or A F along A C, by that Motion the Rectangle or its Area will be produced. Wherefore a Rectangle is rightly ſaid to be produced from the Drawing of two Lines into one another, or the Multiplication of them one by the other. When therefore you have theſe Words, [the Rectangle under (or of) A C, C B,] or for Brevity's ſake, [the Rectangle A C B,] there is meant that Rectangle which is contained under A C and C B multiplied one into the other. In like manner, when we ſay the Rectangle under A B, B C, or the Rectangle A B C, there is deſigned the Rectangle contained under the right Lines A B and B C, multiplied by one another.

Fig. 2. *l.* 2.

Moreover, of Rectangles ſome are oblong, ſome are ſquare. The oblong Rectangle is that which hath its contiguous Sides unequal, or which is contained under two unequal right Lines. The ſquare Rectangle, that which is contained under two equal right Lines.

PROPOSITION I. Theorem.

Fig. 1. *l.* 2.

IF there be two right Lines (A B, A C) one whereof is divided into as many Parts as you will (A E, E F, F C) the Rectangle compriz'd under thoſe two (A B, A C) is equal to all the Rectangles together, which are contain'd under the undivided Line (A B) *and the ſeveral Parts of the divided Line (AE, EF, FC).*

Make AB perpendicular to AC, through B draw the infinite Line BR parallel to AC. From E, F, C, erect the Perpendiculars EI, FL, CQ. BC will be a Rectangle under AB and AC; and is equal to the Rectangles BE, IF, LC; that is, (becauſe as well IE as LF are equal[y] to AB) equal to the Rectangles under AB, AE; AB, EF; AB, FC.

[y] *Per* 29, & 34. *l.* 1.

Scholium.

THE ten firſt Theorems of this Book are true alſo in Numbers, if they as Lines be divided into Parts. The numerical Rectangles are produced from the Multiplication of two Numbers, and the numerical Squares from the Multiplication of the ſame Number by itſelf.

[*Let the undivided Number be* 9, *and the divided one* 12. *The Rectangle which is from* 9 *multiplying* $12=108$ *will be equal to the three Rectangles,* 27, 36, *and* 45, *which are produced from* 9 *multiplied by* 3, *and* 4, *and* 5, *reſpectively and ſeparately. Or let the Number* 432 *be as it were a Multiplicand divided into* 400, *and* 30, *and* 2; *and the Number* 8 *an undivided Multiplier*; $8 \times 432 = 3456$ *will be equal to* $8 \times 400 = 3200 + 8 \times 30 = 240 + 8 \times 2 = 16$. *And from this Propoſition therefore the Demonſtration of Multiplication is to be derived.*

PROP.

PROP. II. Theorem.

Fig. 2. *IF the right Line (AB) be cut any where (as in C), the two Rectangles under the Whole (AB) and the Parts (AC, CB) are equal to the Square of the whole Line (AB).*

Fig. 17. [*For AD is the Square of the Whole; and AH, CD are Rectangles under the Whole AB and the Parts AC, CB.*]

[*Let the Number* 8 *be divided into* 5 *and* 3; *the Square of the Whole* 8 × 8 = 64 *is equal to the Rectangles* 8 × 3 = 24, + 8 × 5 = 40.]

PROP. III. Theorem.

Fig. 3. *LET a right Line as (AB) be cut any where, (as for Inſtance in C); the Rectangle contain'd under the Whole AB, and either of the Parts (BC) is equal to a Rectangle under the Parts (AC, CB), together with the Square of the ſaid Part (BC).*

Fig. 18. [*For AF is the Rectangle under the whole Line AB, and the Part AC; and CF is the Rectangle under the Parts; as AE is the Square of the Part AC.*]

In Numbers. Let the Number 7 *be divided into the Parts* 3 *and* 4. *The Rectangle of* 7 × 3 = 21 *is equal to the Rectangle of* 3×4 = 12, *together with the Square* 3×3 = 9. *In like manner* 7 × 4 = 28 *is equal to the Rectangle* 3×4 = 12 + *the Square* 4×4 = 16.]

PROP.

PROP. IV. Theorem.

Fig. 4.

L*ET a right Line as (FL) be cut any where, as in (O): The Square of the Whole ſhall be equal to the Squares of the Parts (FO, OL) and to two Rectangles contain'd under the Parts (FO, OL).*

Fig. 19.

[*For FD is the Square of the Whole; and CG and CL the Squares of the Parts; and CF, CD, two Rectangles under the Parts.*]

In Numbers. Let the Number 10 *be divided into two Parts* 7 *and* 3. *The Square of* 10 × 10 = 100 *is equal to the Squares of the Parts* 7 × 7 = 49, *and* 3 × 3 = 9, *and to the two Rectangles* 7 × 3 = 21, *and* 7 × 3 = 21. *And on this Propoſition depends the Extraction of the Square Root.*

Fig. 19.

Coroll. (1.) *Hence it is manifeſt, that the Parallelograms about the Diameter of a ſquare, (OI, HK) are Squares.*

(2.) *As likewiſe that the Diameter of every Square biſects the Angles of it.*

(3) *And that the Square of half of the Line is a fourth Part of the Square of the whole Line. For in that Caſe the Rectangles and Squares end in four equal Squares.*]

PROP. V. Theorem.

Fig. 5.

I*F a right Line as (QX) be cut equally in (R), and unequally in (S) the Rectangle contain'd under the unequal Parts (QS, SX) taken together with the Square* *of the intermediate Part (RS) ſhall be equal to the Square of the half (QR).*

Fig. 20. [*For QH is the Rectangle under the unequal Parts, and LG the Square of the intermediate Part, and RF the Square of half the Line; and therefore, becauſe the Rectangle QL is equal to the Rectangle SF, and the reſt of the Space is common to both, the Propoſition is manifeſt.*]

Let the Number 8 *be divided equally, that is, into* 4 *and* 4, *and unequally into* 5 *and* 3. *The Rectangle of* $5 \times 3 = 15$ *together with the Square* $1 \times 1 = 1$ *ſhall be equal to the Square* $4 \times 4 = 16$.]

PROP. VI. Theorem.

Fig. 6. *IF a right Line (AB) be divided into two equal Parts in (C), and to it a certain right Line (BF) be adjoin'd; the Rectangle contain'd under the whole compound Line (AF) and the adjoin'd one (BF) taken together with the Square of half the Line (CB) ſhall be equal to the Square of (CF) which is compounded of half the Line AB and the adjoin'd one.*

Fig. 21. [*For AN is the Rectangle under the whole compound Line, and the adjoin'd one; and KG the Square of half the Line AB; and CE the Square of the Line compounded of half the Line AB, and that which was added. Wherefore becauſe the Rectangle HE is equal to the Rectangle AK, and the reſt of the Space is common to both, AN and KG is equal to CE.* Q. E. D.

[*If the Number* 6 *be divided into the two equal Parts* 3 *and* 3; *and to it be added the Number* 2.

The Rectangle of 8 × 2 = 16, *taken together with the Square* 3 × 3 = 9 *ſhall be equal to the Square* 5 × 5 = 25.]

Coroll. *Hence, with* Maurolycus, *with one ſingle Obſervation we learn to meaſure the Diameter of the Earth. Let the Altitude of the Mountain A D be known, and A B the Line touching* Fig. 22. *the Earth be known by meaſuring. Let the Line D E be cut into two equal Parts in the Centre C, and to it be added the Line A D. Now becauſe the Rectangle under A E, A D, together with the Square of D C, is by this Propoſition equal to the Square of A C, that is, equal to the*[z] *Squares of the Lines A B, B C: From hence it follows, that, if you take away on both Sides the Square of C D or C B, the Rectangle which is under A E, A D is equal to the Square of A B. Therefore let the known Square of A B be divided by the known Altitude of the Mountain A D, and the Quotient will give the Line A E. From which ſubtract the known Altitude of the Mountain A D, the remaining Line D E will be the Diameter of the Earth.* Q. E. I.

[z] *Per* 17. *l.* 1.

PROP. VII. Theorem.

I*F a right Line (A B) be cut any where (as in C), the Square of the whole Line (A B) taken together with the Square of either of the Segments (A C) is equal to two Rectangles contained under the whole (A B), and that Segment (A C) together with the Square of the other Segment (C B).* Fig. 7.

Fig. 23. *[For E B is the Square of the whole Line, and A L the Square of the Part A C. But the two Rectangles under the whole Line and that Part, E I, H L, together with G B the Square of the other Part, poſſeſs the ſame Space that E B and the Square of A C doth. Therefore they are equal to E B and the Square of A C.]*

[Let the Number 13 *be divided into any two Parts, as* 9 *and* 4. *The Square* $13 \times 13 = 169$. *together with that* $9 \times 9 = 81$, *is equal to* $13 \times 9 = 117$, *and* $13 \times 9 = 117$, *and the Square* $4 \times 4 = 16$.]

PROP. VIII. Theorem.

Fig. 8. *IF a right Line (L F) be divided into two equal Parts in (I), and to it a certain right Line be adjoin'd (F O); the Rectangle (L I O) which is contain'd under the half of the Line (L I) and the Line (I O) that is compounded of half the aforeſaid Line, and the Line adjoin'd; this Rectangle taken four Times, together with the Square of the adjoin'd Line (F O), ſhall be equal to the Square of the whole compound Line (L O).*

Fig. 24. *[For A L is the Square of the whole Compound, containing four equal Rectangles under L I and I O (to wit, D R, B Q, R O, and the fourth made up of L R and Q H added together) and with thoſe four Rectangles the Square H E. From whence the Propoſition is manifeſt.]*

[Let the Number 12 *be divided into* 6 *and* 6; *and the Number* 4 *be added to it. The four Rectangles* $10 \times 6 = 240$ *and* $4 \times 4 = 16$ *are equal to the Square* $16 \times 16 = 256$.]

PROP. IX. Theorem.

IF a right Line (AC) be divided equally in (B) and unequally in (F), the Squares of the unequal Parts (AF, FC) will be double to the Squares of half the Line (*AB*), *and of the intermediate Part* (*BF*).

Fig. 9.

[*Let BE be equal and perpendicular to BA. From hence the Conſtruction being made, as the Figure ſhews, the Lines AB, BE, BC will be equal: as alſo the Lines EG, GQ will be equal. The Angles AEC, ABE, CBE, EGQ, QFC, will be equal; and the Angles AEB, BEC, ECA, CQF, EQG half right ones. From whence the Square of AE will be double*[a] *to the Square of AB, which is half of AC, and the Square of EQ double to the Square of GQ or BF the intermediate Line. But the Squares of AE and EQ are*[b] *equal to the Square of AQ, that is, to the Squares of AF and FQ or FC the unequal Parts.* Q. E. D.]

Fig. 25.

[a] *Per* 47. *l.* 1.

[b] By the ſame.

[*Let the Number* 32 *be divided equally into* 16 *and* 16, *and unequally into* 20 *and* 12. *The Square* $20 \times 20 = 400$ *with the Square* $12 \times 12 = 144$, *are double to the Squares of* $16 \times 16 = 256$ *and* $4 \times 4 = 16$.]

PROP. X. Theorem.

IF a right Line (*FI*) *be divided into two equal Parts in* (*L*), *and to it a certain right Line* (*as IO*) *be adjoin'd; the Square of the whole compound Line,* (*FO*) *taken together with the Square of the additional Line* (*IO*),

Fig. 10.

(IO), ſhall be double to the Squares, which are deſcribed upon the half Line (F L), and (L O) that which is compounded of half the Line (F I) and the additional Line.

Fig. 26. [*For a Conſtruction being ſuppoſed not unlike to the former; the Square of F E is double to the Square*[c] *of the half Line F L, and the Square of E G is double to the*[c] *Square of E Q or L O which is compounded of the half Line and the additional one. But the Squares of F E and E G are*[c] *equal to the Square of F G; that is, to the Square of F O the whole compound Line; taken together with the Square of O G or O I the additional Line.* Q. E. D.]

[c] *Per* 47. *l.* 1.

[*Let the Number* 40 *be divided into* 20 *and* 20, *and to it let there be added the Number* 14. *The Square* 54 × 54 = 2916, *with the Square* 14 × 14 = 196, *are double to the Square of* 20 × 20 = 400 *taken together with* 34 × 34 = 1156.]

PROP. XI. Problem.

Fig. 11. S*O to cut the given right Line (AB) in (C) that the Rectangle (ABC) which is contain'd under the whole Line and one Part, ſhall be equal to the Square of the other Part (AC).*

From A erect a Perpendicular AF equal to AB. Biſect AF in X. Draw the right Line XB; from the Line FA drawn forth cut off XI equal to XB. Then cut off AC equal to AI. I ſay the thing is done.

For let the Square B A F S be perfected; and a Perpendicular being drawn through C, let the Rectangle F I L O be perfected alſo. Becauſe F A is biſected in X, and to it is added A I; there ſhall be

$$\left\{\begin{array}{l}\text{the Rect. FIA}\\ \quad + \\ \text{Square of XA}\end{array}\right. =^{d} \text{to the Square of XI}$$

$$\text{That is,} = \text{to the Square of XB}^{e}$$

$$\text{That is,} = \text{to the Squares of} \left.\begin{array}{r}\text{BA}\\ \text{AX}\end{array}\right\}^{f}$$

Therefore let there be taken away on both Sides the Square of X A; there will remain

Rectangle F I A or F L.

= A S the Square of the Line B A;

Wherefore again, the common Rectangle A O being taken away,

A L will remain equal to C S.

But A L is the Square of the Line A C, ſeeing by the Conſtruction A C and A I are equal. And C S is the Rectangle A B C, foraſmuch as B S is equal to A B. Therefore the Rectangle A B C is equal to the Square of A C. Therefore we have cut the Line A B, as it was required.

[d] *Per* 6. *l.* 2.

[e] By the Conſtruction.

[f] *Per* 47. *l.* 1.

Scholium.

THE ten firſt Propoſitions of this Book are true alſo in Numbers. But this eleventh cannot be exemplify'd in Numbers; for no Number can be ſo divided that the Product of the Whole multiplied by one Part ſhall be equal to the Square of the other. The Force of this Section of a Line is wonderful, for which ſee *Prop.* 30. *Lib.* 6.

PROP. XII. Theorem.

Fig. 12. *IN an obtuſe-angled Triangle (ACB), the Square of the Side (AB) oppoſite to the obtuſe-Angle (C), exceeds the Squares of the other Sides (AC, CB), by the Rectangle (BCF) twice taken; which ſame Rectangle is comprized under (BC), one of the Sides containing the obtuſe Angle, and the Line (CF) which is intercepted betwixt the Perpendicular (AF) and the obtuſe Angle.*

The Square of AB is equal to the Squares

[g] *Per* 47. *l.* 1. of $\left\{\begin{array}{l} AF \\ BF \end{array}\right\}$[g]

But the Square of BF is equal to the Squares of FC, CB, with the Rectangle FCB twice

[h] *Per* 4. *l.* 2. taken[h]. Therefore if you ſubſtitute theſe for the Square of BF; then

$$\text{the Square of AB is equal to } \left.\begin{array}{r} \text{AF Square} \\ \text{FC Square} \\ \text{CB Square} \\ \text{and Rectangle BCF twice.} \end{array}\right\}$$

[i] *Per* 47. *l.* 1. But the Squares of AF, FC are[i] equal to the Square of AC. Wherefore this being ſubſtituted for them,

$$\text{AB Square is equal to } \left.\begin{array}{r} \text{AC Square} \\ \text{CB Square} \\ + \text{ Rectang. BCF twice.} \end{array}\right\}$$

PROP. XIII. Theorem.

Fig. 13, 14. *IN any Triangle whatſoever (as ACB) the Square of the Side (AB) oppoſite to an acute Angle (C) is exceeded by the Squares of the*

the other Sides (AC, CB) by the Rectangle (BCF) twice taken; which ſame Rectangle is contain'd under (BC) one of the Sides comprehending the acute Angle (C); and the Line (FC) which is intercepted betwixt the Perpendicular (AF) let fall upon the Side (BC) from its oppoſite Angle (A), and the acute Angle (C).

The Square of BC is equal to[k] the Rect. (BFC twice, + FC Square + FB Square [k] *Per* 4. *l.* 2.

And AC Square is equal to[l] CF Square + FA Square [l] *Per* 47. *l.* 1.

Wherefore the two together {BC Squ. + AC Squ.} are equal to Rect. (BFC twice, + BF Square + 2 FC Square + AF Square

But the Rectangle BFC twice, together with the Square of FC twice, is[m] equal to the Rectangle BCF twice. Therefore this being ſubſtituted for them, [n] *Per* 3. *l.* 2.

BC Squ. + AC Squ. are equal to the Rect. BCF (twice + BF Square + AF Square

But the Squares of AF, BF are equal to[n] the Square of AB. Therefore this being ſubſtituted for them, [n] *Per* 47. *l.* 1.

BC Squ. + AC Squ. are equal to the Rect. BCF twice + AB Squ.

That is, BC Square + AC Square do exceed AB Square by the Rectangle BCF twice taken.

Corollary.

Fig. 15. THE Propoſition is true, although the Perpendicular falleth without the Triangle. And the Demonſtration is almoſt the ſame.

° Per 12. l. 2. [*More briefly thus:* $ACq =$ ° $ABq + CBq + 2\,CBF$. *Add on both Sides* CBq, *then* $ACq + CBq = ABq + 2\,CBq + 2\,CBF =$ p $ABq + 2\,BCF$. Q. E. D.]

p Per 3. l. 2.

Scholium.

FROM this Propoſition and the 47th of the former Book, we have the Meaſure of any Triangle whatſoever, whoſe three Sides are known, although the Area be altogether inacceſſible. For, by the Help of theſe Theorems, the Perpendicular is known, albeit the Impediments of the Place ſhould not permit us to mark it out. But note, That the Perpendicular, multiplied by half the Side on which it falls, produceth the Area of the Triangle, as appears out of the Scholium of Propoſition 41. Lib. 1.

Fig. 13, or 14. Let there be any Triangle (as ABC) having its Sides known. It is required to give the Perpendicular AF, which falls from the given Angle A upon the oppoſite Side CB.

Take the Square of this Side AB oppoſite to the acute Angle C, out of the Sum of the Squares of AC and BC. By the 13th the Remainder ſhall be the Rectangle BCF twice taken. Divide half of the Remainder, that is, the Rectangle BCF by the known Side BC; thence will ariſe the right Line CF. Take the Square of the right Line CF out of the

Square of A C. The Remainder will give[q] the Square of A F, whoſe ſquare Root will give the Perpendicular A F.

This Thing alſo may be obtained out of Propoſition 12. But the 13th ſufficeth, foraſmuch as in every Triangle the Perpendicular, let fall from ſome one of the Angles unto the oppoſite Side, falls within the Triangle.

PROP. XIV. Problem.

THE right-lin'd Figure (Q X Z) being given, to make a Square equal to it. Fig. 16.

Make[r] a Rectangular Parallelogram C I equal to the Rectilinear Q X Z; the Sides of which Parallelogram, if they ſhall be equal, you have already made the Square which was required; if they be unequal draw forth the greater Side I A unto L, until A L ſhall be equal to A C. Then biſect I L in Z; from which as from a Centre through I and L deſcribe a Circle, and let C A be produced till it meets the Circumference in B. The Square of the right Line A B is equal to the given Rectangle Q X Z.

For let the right Line Z B be drawn; becauſe I L is cut equally in Z and unequally in A; the

$$\left.\begin{matrix}\text{Rectangle } IAL \\ +\ ZA \text{ Square}\end{matrix}\right\}\text{are equal}^{s}\text{ to } ZL \text{ Squ. that is,}$$

$$\text{equal to}^{t}\ ZB \text{ Square, that is,}$$

$$\text{equal to}^{u}\ \left.\begin{matrix}ZA \text{ Square} \\ +\ AB \text{ Square.}\end{matrix}\right\}$$

Taking

[q] *Per Prob.* 2. *Schol. Poſt.* 47. *Lib.* 1.

[r] *Per* 45. *l.* 1.

[s] *Per* 5. *l.* 2.

[t] By the Conſtruction.

[u] *Per* 47. *l.* 1.

Taking away therefore on both Sides the common Z A q, there remains

Rect. I A L equal to A B q; that is,

Becauſe A C and A L are equal, the Rectangle C I equal to A B Square, and conſequently A B Square equal to the Rectilineal [w] Q X Z.

[w] By the Conſtruction.

Scholium.

EUCLID's Conſtruction of this Problem requires, that the given Rectilineal be reduced unto a Rectangle by *Prop.* 45. *l.* 1. Which Reduction being operoſe enough, the Problem perhaps will more readily be diſpatch'd after this Manner:

Let the given Rectilinear be reſolv'd into as many Quadrangles (X, Z) as it can. Then to each Quadrangle [x] make an equal Rectangle. If there remain, as here it happens, one Triangle (Q), to it alſo [y] make a Rectangle equal. Then to each Rectangle by this 14 *l.* 2. make an equal Square; and laſtly to all theſe Squares let one equal one be made [z]. This will be equal to the given Rectilinear Q X Z.

[x] *Per Schol. p.* 45. *l.* 1.

[y] *Per Coroll. p.* 42. *l.* 1.

[z] *Per Prob.* 1. *Schol. p.* 47. *l.* 1.

THE ELEMENTS OF EUCLID.

BOOK III.

THE fundamental Properties of the moſt perfect amongſt plain Figures are demonſtrated in this Book, The Uſefulneſs of the Book is manifeſt by this one Thing alone, that it treats of a Circle, that abundant Source of admirable Things through the whole Mathematicks. The more famous Theorems are 16, 20, 21, 22, 31, 32, 35, 36.

DEFINITIONS.

1. THOSE Circles are equal whoſe Diameters or Semidiameters are equal.

2. A right Line (FB) is ſaid to touch a Circle, when it doth ſo meet it in the Point (B), that albeit it be produced it doth not cut it. Fig. 20. l. 3.

3. Circles

Fig. 13, 14. 3. Circles are ſaid to touch one another, when they do ſo meet that they do not cut each other.

Fig. 18. 4. In a Circle the right Lines (B C, F L) are ſaid to be equi-diſtant from the Centre (A), when the Perpendiculars which are let fall upon them from the Centre (A O, A I) are equal.

Fig. 37. 5. Segments or Portions of a Circle are the Parts into which the right Line (C E which cuts the Circle) doth divide it.

Fig. 33. 6. An Angle in a Segment is that (B Q C) which is contain'd under the right Lines, which are drawn unto one Point of the Circumference (Q) from the Ends of the Segment, (B, C).

Fig. 33. 7. The Angle (C Q B is ſaid to ſtand upon the Circumference (B O C), as being oppoſite to it.

Fig. 11. 8. A Sector is that Part of a Circle which is contain'd by two Semidiameters, as (A B, A F), and an Arch as (B F or B C F) intercepted betwixt the Semidiameters.

PROPOSITION I. Problem.

Fig. 1. *l.* 3. *TO find the Centre of a given Circle.*

Let the right Line (B C) be drawn in the Circle at random, which biſect in Q. Through Q draw the Perpendicular L F, which biſect in A. A ſhall be the Centre.

If you deny it; let the Centre be O, which is without the right Line F L (for in F L it cannot be, foraſmuch as this Line is divided every where unequally but in A): and let there be drawn B O, Q O, C O. Becauſe therefore you ſuppoſe O to be the Centre, B O, C O muſt be equal; and the Triangles BOQ, COQ, muſt be equilateral to each other; ſeeing by the Conſtruction BQ and CQ are equal, and QO is common. Therefore the Angle OQC [a] is equal to the Angle OQB. Therefore OQC is a right Angle [b], and conſequently equal to LQC which is a right one by Conſtruction, a Part to the Whole. Which is abſurd.

[a] *Per* 8. *l.* 1.
[b] *Per Def.* 14. *l.* 1.

Corollary.

FROM what hath been demonſtrated it appears, that, if the right Line (LF) cuts another right Line BC into two equal Parts and perpendicularly, the Center is in the Line that cuts the other.

The Centre of a Circle is very eaſily found by a Square; the Top of it (Q) being applied to the Circumference; for if the right Line DE, joining the Points D and E in which the Sides of the Square cut the Circumference, be biſected in A, (A) ſhall be the Centre. The Demonſtration whereof depends on *Prop.* 31. *l.* 3. Fig. 2.

PROP. II. Theorem.

IF in the Circumference of a Circle there be taken two Points (B and C) the right Line which is drawn through them falls entirely within the Circle. Fig. 2.

Let there be taken in the Line BC any Point whatſoever, as O, and from the Centre A, be drawn AB, AO, AC. Becauſe AB, AC

A C are equal, the Angles alſo B and C are [c] equal. Becauſe therefore A O C is [d] greater than the internal one B, it ſhall be greater alſo than C. In the Triangle therefore O A C, the Side A C, ſubtending the greater Angle A O C, is [e] greater than the Side A O ſubtending the leſſer Angle C. Seeing therefore A C reaches no farther than from the Centre to the Circumference, A O ſhall not reach ſo far. Therefore the Point O ſhall fall within the Circle. The ſame thing may be ſhew'd of any other Point of the Line B C. Therefore B C falls wholly within the Circle.

[c] *Per* 5. *l.* 1.

[d] *Per Coroll.* 1. *Prop.* 32. *l.* 1.

[e] *Per* 19. *l.* 1.

The Propoſition is alſo manifeſt from the very Notion of a right Line and a Circle.

Coroll. *Hence it follows, that a right Line, touching a Circle, toucheth it in one ſingle Point only. For, if it touched the Circumference in two Points, it would be a right Line drawn through two Points of the Circle, and conſequently would fall within the Circle, contrary to the Definition of a Tangent. And by the like Reaſoning (in paſſing from Planes to Solids) it might be prov'd, that every Plane toucheth a Sphere only in one Point.*

PROP. III. Theorem.

Fig. 3.

IF in a Circle a right Line (B L) drawn through the Centre biſects another (C F) not drawn through the Centre, it will cut it perpendicularly. And, if it cut it perpendicularly, it will biſect it.

Part 1. From the Centre (A) let there be drawn A C, A F. The Triangles X and Z are equilateral to each other. For C O, F O are by the Hypotheſis equal, and AC, AF are ſo, becauſe drawn from the Centre; while AO is common to both. Therefore the Angles AOC, AOF are [f] equal. Therefore right [g] ones. Which was the firſt Part.

[f] *Per* 8. *l.* 1.

[g] *Per Def.* 14. *l.* 1.

Part 2. Becauſe by the Hypotheſis A O C, A O F are equal Angles; A C Square ſhall [h] be equal to the Squares of A O, O C together; and A F Square equal to the Squares of A O, O F together. Seeing therefore the Squares of A C, A F are equal, the Squares of A O, O C together will be alſo equal to the Squares of A O, O F together: Wherefore taking away the common Square A O, the Squares of O C, O F remain equal. And therefore the right Lines O C, O F are equal. Which was the other Part.

[h] *Per* 47. *l.* 1.

Coroll. (1.) *Hence in every equilateral Triangle, and in that alſo which is only an* Iſoſceles, *a Line which, falling from the Top of the Angle, biſects the Baſe, is perpendicular to it. And on the contrary, a Line, which falling from the Top of the Angle is perpendicular to the Baſe, doth biſect it.*

(2.) *Hence it follows, that half of the Chord of every Arch is the right Sine of half the Arch.*

PROP. IV. Theorem.

Fig. 4, 5.

I*F in a Circle two right Lines (BC, FL), not drawn both of them through the Centre, cut each other, they cannot biſect each the other.*

For

Fig. 5. For, if one of them L F paſſeth through the Centre, it is manifeſt that it ſhall not be biſected by B C which doth not paſs through the Centre.

Fig. 4. If neither of them paſſes through the Centre, from the Centre A draw A O. If now B C, F L were both biſected in O, the Angles A O C, A O L would [i] be right Angles, and conſequently equal; the Whole to a Part, which is abſurd.

[i] By the foregoing.

PROP. V, VI. Theorem.

Fig. 6, 7. *Circles cutting each other, or inwardly touching one the other, have not the ſame Centre.*

For, if it were otherwiſe, the right Lines A B, A C F, drawn from the common Centre A, would be equal; and A C would be equal to A F; a Part to the Whole, becauſe they are both equal to A B. Which is abſurd.

PROP. VII. Theorem.

Fig. 8. *If in a Circle there be taken any Point beſides the Centre (A), as the Point (C), and divers right Lines fall from thence unto the Circumference (as CB, CL, CO, CF);*

1. *(CB) which paſſeth through the Centre will be the greateſt.*

2. *The remaining Part of the Diameter (CF) will be leaſt.*

3. *Of the reſt that will be the greater, which is nearer to the greateſt.*

4. *And no more than two equal Lines can be drawn from the ſaid Point (C) which is different from the Centre, unto the Circumference.*

Part 1. Let AL be drawn from the Centre A. Becauſe AL, AB are equal, the common Line AC being added to each. AC and AL together are equal to CB. But AL + AC are greater than LC[k]. Therefore CB is greater than LC. In the ſame manner BC will be ſhew'd to be greater than any other.

Part 2. From the Centre A draw AO. AO (that is, AF) is leſs than AC, CO[l]. Therefore taking away the common Line AC, CO remains greater than CF. In the ſame manner CF is prov'd to be leſs than CQ, or any other.

Part 3. In the Triangles COA, CLA, the Sides LA, AC, are equal to OA, AC, each to each. But the Angle LAC is greater than the Angle OAC. Therefore[m] the Baſe LC is greater than the Baſe OC.

Part 4. This is manifeſt from what goes before. For if there could be three drawn equal, CO, CI, CQ, there would be two on the ſame Side equal: Which is contrary to Part 3.

Coroll. *By the like Reaſoning,* Theodoſius *gathered, that of the Arches of great Circles drawn upon the Surface of a Sphere, from any Point diverſe from the Pole of a certain Circle, unto that Circle, the greateſt is that which paſſeth through the Pole of that Circle; the leaſt, that which is*

drawn

[k] *Per* 20. *l.* 1.

[l] By the ſame.

[m] *Per* 24. *l.* 1.

drawn unto the oppoſite Point; and of the reſt, that is the greater which is neareſt to the greateſt; as alſo that no more than two equal Arches can be drawn from that Point unto the Circle. And in the like manner may the Reader reaſon of himſelf on ſome other of the Propoſitions of this Book; it being very eaſy to paſs from Planes to Solids in theſe Argumentations.

PROP. VIII. Theorem.

Fig. 9, 10. I*F from a Point* (*A*) *taken without a Circle, there be drawn unto the Circle the right Lines* (*AB, AC, AF*) *or* (*AO, AQ, AR*);

1. *Of thoſe which fall upon the concave Circumference, the greateſt is* (*AB*) *which paſſes through the Centre* (*Z*).
2. *Of the reſt, that is the greater, which is nearer to the greateſt* (*AB*).
3. *Of thoſe which fall without the Circle or upon the convex Periphery, the leaſt is* (*AO*) *which being produced would paſs thro' the Centre* (*Z*).
4. *Of the reſt, that which is nearer to the leaſt is leſs than that which is farther off.*
5. *No more than two equal Lines can be drawn unto the Circumference from the ſame Point* (*A*), *whether they fall within the Circle, or only without.*

Fig. 9. Part 1. From Centre Z draw ZC; becauſe ZC, ZB are equal, the common AZ being added to each, AZ, + ZC are equal to AB.

But AZ+ZC are [n] greater than AC. Therefore AB is greater than AC. In like manner AB will be ſhewed to be greater than any other whatſoever. [n] *Per* 20. *l.* 1.

Part 2. Draw ZF. Becauſe in the Triangles AZC, AZF, the Sides AZ, ZC are equal to AZ, ZF is each to each; but the Angle AZC is greater than AZF, therefore the Baſe AC [o] will be greater than the Baſe AF. [o] *Per* 24. *l.* 1.

Part 3. Draw ZQ. The two Lines AQ, QZ are greater than AZ [p]. Taking away therefore the Equals ZQ, ZO, there remains AQ greater than AO. In the ſame manner AO is proved leſs than any other. Fig. 10. [p] *Per* 20. *l.* 1.

Part 4. Draw ZR. The right Lines AQ, QZ are leſs than AR, RZ [q]; therefore the Equals, ZQ, ZR being taken away, AR remains greater than AQ. [q] *Per* 21. *l.* 1.

Part 5. This is manifeſt from the four foregoing.

PROP. IX. Theorem.

IF from ſome Point within a Circle (as A) more than two equal right Lines can be drawn unto the Circumference; that Point is the Centre. Fig. 11.

This is manifeſt from Part 4. of Propoſition 7.

PROP. X. Theorem.

CIrcles cut each other in two Points only. Fig. 12.

For let them cut, if it may be, in more (B, C, F.) From A the Centre of the Circle LQ, let there be drawn to the Points, B, C, F, the Lines, AB, AC, AF: theſe will be equal. Becauſe therefore, from the Point A within the Circle OS, there are drawn three equal Lines AB, AC, AF, unto its Circumference, A muſt alſo be the Centre [r] of the Circle OS. Therefore the Circles LQ, OS, which cut one another, have the ſame Centre. Which contradicts Propoſition 5.

[r] By the foregoing.

PROP. XI. Theorem.

Fig. 13.

IF two Circles touch each other inwardly, a right Line drawn through their Centres (A and I) paſſes through the Point of Contact (B).

If you deny it, let the Centres have, if it may be, that Situation that a right Line paſſing through them ſhall fall without the Contact B, cutting the Circles (in O and L); let the Centre be A and C; and join AB, CB. Becauſe therefore CB, CO are equal, the common AC being added to each of them, AC, + CB ſhall be equal to AO. But AC, CB are [s] greater than AB, that is, than AL [t]. Therefore alſo AO is greater than AL, a Part than the Whole. Which is abſurd.

[s] *Per* 20. *l.* 1.
[t] By the Definition of a Circle.

PROP. XII. Theorem.

Fig. 14.

IF Circles touch one another on the Outſide, a right Line which joins the Centres muſt paſs through the Point of Contact.

If it be denied, let the Centres be ſo plac'd, as for Inſtance in A and B, that the Line paſſing through them ſhall not paſs through the Contact S, but cut the Circles in O and Q. Let the Points A, S, and B, S be joined. Then A S, B S together will [u] be greater than A B. But A S is [w] equal to A O, and B S equal to B Q. Therefore A O and B Q together will be greater than A B, a Part than the Whole. Which cannot be.

[u] Per 20. l. 1.

[w] By the Definition of a Circle.

[Coroll. *A right Line, drawn from the Centre of one of the Circles through the Point of Contact, will paſs through the Centre of the other.*]

PROP. XIII. Theorem.

CIRCLES touch both one another, and a right Line, in a Point only. Fig 15, & 16.

Fig. 15. For let two Circles touch one another inwardly in a Part of the Circumference L C, if it may be. Then a right Line drawn thro' the Centres A and B will [x] paſs through the Point of Contact, as in C. Let there be drawn alſo A L, B L. Becauſe therefore B L, B C are equal (for they are drawn from the Centre B unto the Circumference O L C) the common Line A B being added, A B, B L ſhall be equal to A C. But A C is equal to A L, for they are both drawn from the Centre A unto the Circumference L Q C. Therefore A B, B L are equal to A L, contrary to *Prop.* 20. *l.* 1.

[x] Per 11. l. 3.

Fig. 16. Then let the two Circles touch one another on the Outſide, in the Arch O L, if it may be. The right Line A P joining the Centres

[a] *Per* 12. *l.* 3.

will paſs through the Point of Contact[a], as in O for Inſtance. Let A L, P L, be drawn. The two Sides of the Triangle A L, P L will be equal to A O, P O or the whole A P; contrary to *Prop* 20. *l.* 1.

[a] *Per Corol.* 11. *Prop.* 32. *l.* 1.
[b] *Per Corol.* 3. *Prop.* 32. *l.* 1.

Laſtly, let the right Line B F and the Circle touch each other, if it may be, in ſome Part (C E): Let there be drawn unto the Centre the right Lines C A, E A. The Lines C A, E A will then be equal: And therefore the Triangle C A E is an *Iſoſceles.* Wherefore the Angles C and E[a] are acute. And therefore a Perpendicular let fall unto B F from the Centre A will fall betwixt E and C[b], as for Inſtance in D. There will therefore both A C and A E be equal to the Perpendicular A D, which is abſurd, and contrary both to *Coroll.* 14. *p.* 32. and to *Prop.* 47. *l.* 1.

Corollary.

Fig. 17.

CIrcles whoſe Centres are in the ſame right Line, and which cut it in the ſame Point B, do touch one another in that Point only.

This Propoſition is manifeſt from the very Notion of the Lines which are compar'd together. For neither can a right Line and the curve Circumference of a Circle, or the diverſe Curvatures of unequal Circumferences, or two Curvatures both convex, agree as to any Part of themſelves. But they would agree, if they touched one another in ſome entire and proper Part.

PROP.

PROP. XIV. Theorem.

IN a Circle equal right Lines (BC, LF) are equally diſtant from the Centre (A). And what Lines are equi-diſtant from the Centre are equal. Fig. 18.

From the Centre (A) let there be drawn (AC, AF). Likewiſe (AO, AI) at right Angles to BC, FL. Thus BC, FL ſhall be biſected[c] in O and I.

[c] *Per* 3. *l.* 3.

Seeing therefore the whole Lines BC, FL are ſuppoſed equal, the Halves alſo OC, IF muſt be equal, and conſequently the Squares of them are alſo equal. Seeing therefore the Squares of AC, AF are equal, and the Square of AC is equal to OCq and OAq, as alſo the Square of AF is equal to IFq, and IAq[d]: It follows that the two Squares OCq, OAq are equal to the two Squares IFq, IAq. Wherefore taking away the Squares of OC, IF (which before were ſhew'd to be equal) the Square of OA remains equal to the Square of AI. Therefore the Perpendiculars OA, AI, are equal. Therefore[e] BC, FL are equi-diſtant from the Centre. Which was the firſt Part. Then for the Converſe of it;

[d] *Per* 47. *l.* 1.

[e] *Per Def.* 4. *l.* 1.

If the Diſtances OA, AI are ſuppoſed equal, then the Squares of the equal right Lines being taken away, by the ſame Ratiocination it will be ſhew'd that the remaining Squares OCq, IFq are equal, and conſequently that the right Lines OC, IF are equal, which ſeeing they are[f] Halves of the right Lines BC, FL, theſe alſo muſt be equal. Which was the ſecond Part.

[f] *Per.* 3. *l.* 3.

PROP. XV. Theorem.

Fig. 19.

Of right Lines inſcribed in a Circle, the greateſt is the Diameter; and of the reſt, that is the greateſt, which is the neareſt to the Centre.

Let there be any Line, as R S different from the Diameter F L. From the Centre draw A R, A S. The two A R, A S, which are equal to the Diameter, are[g] greater than R S, Therefore, *&c.*

Then let B I be nearer to the Centre than X Z. From the Centre unto them draw the Perpendiculars A C, A Q. A Q ſhall be greater[h] than A C. Take therefore A O equal to A C, and through O draw R S perpendicular to A O, which[i] will be equal to B I; and let A R, A S, A X, A Z be join'd. Becauſe therefore A is the Centre, the Sides A R, A S ſhall be equal to A X, A Z. But the Angle R A S is greater than the Angle X A Z. Therefore the Baſe R S, that is, B I, is greater than the Baſe X Z[k]. *Q. E. D.*

PROP. XVI. Theorem.

Fig. 20.

A Right Line (IF) which being drawn through the Point (B), the Extremity of the Diameter (CB) is perpendicular thereto, falleth all of it without the Circle, and toucheth it in (B). Neither can there any right Line be drawn betwixt itſelf and the Circle unto the Point of Contact (B) but it ſhall cut the Circle.

Part

[g] *Per* 20. *l.* 1.

[h] *Per Def.* 4. *l.* 3.

[i] By the foregoing.

[k] *Per* 24. *l.* 1.

Part 1. Let there be taken in the Line IBF any Point L, unto which from the Centre A draw the Line AL. Becauſe in Triangle ABL, the Angle ABL is a right one, by the Hypotheſis, ALB ſhall be acute [l]. Therefore AL which is oppoſite to the greater Angle B will be greater than AB which is oppoſite to the leſſer Angle L [m]. But AB reacheth only to the Circumference. Therefore AL ſhall reach beyond the Circumference; and conſequently fall without the Circle. Which was the firſt Part.

[l] *Per Corol.* 5. *p.* 32. *l.* 1.

[m] *Per* 19. *l.* 1.

Part 2. Below BF, if it may be, let RB fall wholly without the Circle. Becauſe FBA is a right Angle, by the Hypotheſis RBA will be acute, and therefore AB is not perpendicular to BR. Therefore let there be drawn from the Centre A to BR the Perpendicular AO, which [n] will fall towards R and cut the Circle in Q. Therefore AB which is oppoſite to the greater Angle, AOB, is greater than AO, which is oppoſite to the leſſer, to wit, the acute Angle OBA. But AB is equal to AQ: Therefore AQ alſo is greater than AO, a Part than the Whole.

[n] *Per Corol.* 3. *Prop.* 32. *l.* 1.

Corollary.

1. HENCE it appears again, that the Contact of a right Line and a circular one is only in one Point. Fig. 20.

2. If, from Centres taken in the ſame right Line infinitely protracted, there be deſcribed thro' B infinite Circles, as well leſſer than the firſt BSC, as greater; they ſhall all touch the right Line IF in the ſame one Point B. Fig. 17.

3. Circles therefore, growing into an Amplitude greater that any given one, approach always, even unto Infinity, nearer and nearer to the Tangent, but are never join'd to it, otherwiſe than in one ſingle Point of Contact; which thing, although it be moſt evident, is yet truly admirable.

Fig. 17. 4. From theſe Things it is manifeſt, that every geometrical Line whatſoever is infinitely diviſible. For let there be drawn from ſome Point of the Diameter unto the Tangent the right Line A Q. Infinite Circles, having Centres in the right Line B A infinitely produced, touch the right Line I F by *Coroll.* 2. of this, and one another by *Coroll. p.* 13. in one and the ſame Point B, and conſequently are no where join'd either amongſt themſelves, or with the right Line I F, but in the Point B only. Therefore it is neceſſary that they divide the right Line A Q into infinite Parts, that is, into Parts exceeding any Number aſſignable.

Fig. 20. 5. The Angle of Contingence or Contact L B Q, (that, to wit, which is contained under the Tangent and the Circumference) cannot be divided by any right Line.

Fig. 17. 6. Nevertheleſs, by Circumferences touching the Line I F in the ſame Point, it may be divided and diminiſhed infinitely. And in this and the third Corollary lies hid the whole Myſtery of Aſymptotes, that is, of a right Line approaching unto an Hyperbola, together with itſelf infinitely produced, unto a Diſtance leſs than any given one, yet never concurring with it.

PROP.

PROP. XVII. Problem.

FROM the given Point (B) to draw a right Line which ſhall touch a given Circle (O Q). Fig. 26.

From A the Centre of the given Circle let there be drawn unto the Point B the right Line A B, cutting the Periphery in O. From the Centre A deſcribe through B another Circle B C, and from O draw O C perpendicular to A B, which may meet the other Circle in C. Draw C A meeting the Circle O Q in I. The right Line drawn from B unto I will touch the Circle O Q.

For becauſe the Sides B A, I A, are equal to the Sides C A, A O and the Angle A contain'd betwixt the equal Sides is common to both. In the Triangles I A B, O A C the Angles A O C, A I B are alſo [n] equal. Therefore A I B is a right Angle. For A O C is a right one by the Conſtruction. Therefore B I [o] toucheth the Circle in I.

[n] *Per* 4. *l.* 1.

[o] *Per* 16. *l.* 3.

Scholium.

BY the 31ſt following, from the given Point O a Line touching a given Circle (B Q) may be well drawn thus: Fig. 27.

Let the right Line joining the given Point O and the Centre A be biſected in P. Then from the Centre P through A and O deſcribe a Circle, meeting the given one in B. The right Line O B will touch the Circle.

For A B being join'd, the Angle A B O in the Semicircle is a right one by *Prop.* 31. Therefore by *Prop.* 16. O B toucheth the Circle B Q.

PROP. XVIII. Theorem.

Fig. 28.

IF a right Line (C L) touch a Circle, a right Line (A B) drawn from the Centre (A) unto the Point of Contact (B) is perpendicular to the Tangent.

If it be denied, let ſome other right Line (as A F) be the Perpendicular from the Centre A. This will cut the Circle in O. Becauſe therefore the Angle A F B is ſuppoſed to be a right one, A B F[p] muſt be acute. Therefore A B (that is, A O) is greater than A F[q]; a Part than the Whole.

[p] *Per Corol.* 5. *p.* 32. *l.* 1.

[q] *Per* 19. *l.* 1.

PROP. XIX. Theorem.

Fig. 29.

IF a Line (B C) touch the Circle, and from the Point of Contact (A) there be rais'd (A I) perpendicular to the Tangent, the Centre will be in that Perpendicular.

If you deny it, let the Centre be without A I in Z; and from it let there be drawn unto the Contact the Line Z A. The Angle Z A C will be a right one[r], and therefore equal to the Angle I A C, which by the Hypotheſis is a right one; that is, the Part will be equal to the Whole.

[r] By the foregoing.

PROP. XX. Theorem.

Fig. 30, 31, 32.

THE Angle at the Centre (BAC) is double to the Angle (BFC) which is at the Circumference, when the ſame Arch (BC) is the Baſe of the Angles.

Fig. 30.

Here are three Caſes. In the firſt Caſe, the Sides BA, BF coincide. And then becauſe AF, AC drawn from the Centre are equal, there will be in the Triangle Z, the Angles F and C equal[s]. But BAC is equal to the two Angles F and C[t]. Therefore BAC is double of F.

Fig. 31.

In the ſecond Caſe BA, CA fall within BF, CF, and then FAX being drawn, XAB by the firſt Caſe is double of XFB; and XAC double of XFC. Therefore the whole BAC is double of the whole BFC.

Fig. 32.

In the third Caſe, BF cuts AC, and the Angle BAC is without the Triangle BFC. Here let FAL be drawn. By the firſt Caſe the whole LAC is double of the whole LFC, and LAB taken away is double of LFB taken away. Therefore the remaining Angle BAC is alſo double of the remaining one BFC. *Q. E. D.*

Fig. 53.

[Corollary, *Hence we gather that the Sides of every Triangle are to each other as the Sines of the Angles oppoſite to thoſe Sides reſpectively. Let EFG be any Triangle; about which let a Circle be underſtood to be circumſcrib'd*[u], *and from the Centre of the Circle, let there be let down the Perpendiculars AB, AC, AD, which will*[w] *biſect the Subtenſes. Now as EF is to EG, ſo* $\frac{1}{2}$ *EF*

(that

[s] Per 5. l. 1.

[t] Per 32. l. 1.

[u] Per 5. l. 4.

[w] Per 3. l. 3.

(that is, E B) to $\frac{1}{2}$ *E G (that is, E D). But E B is the Sine of the Angle* [x] *B A E, that is, of half the Angle E A F, that is, of the whole Angle E G F* [y] *oppoſite to the Side E F; and E D is the Sine of the Angle E A D, that is, of half the Angle E A G, that is, of the whole Angle E F G, which is oppoſite to the Side E G. Therefore E F is to E G, as the Sine of the Angle E G F is to the Sine of the Angle E F G.* Q. E. D. *And from this one Propoſition a great Part of Trigonometry is deduced. Which Thing will be worth our Obſervation.*

[x] *Per Corol.* 2. *p.* 3. *l.* 3.
[y] *Per* 20. *l.* 3.

Fig. 86. *l.* 1.

Coroll. (2.) *From the former Corollary we learn to meaſure the Diſtance of the Moon. For aſtronomical Obſervations giving us the Angle of the diurnal Parallax* [z] *B C A, we find out the Diſtance of the Moon by the following Proportion. As the Sine of the Angle A C B is to the Sine of the Angle A B C; ſo is the Semidiameter of the Earth B A, unto the Moon's Diſtance, A C.* Q. E. I.

[z] *Corol.* 16. *p.* 32. *l.* 1.

Fig. 54.

Coroll. (3.) *From the ſecond Corollary we learn alſo to meaſure the Diſtance of the Sun. For there being given by aſtronomical Obſervations the Angle of the menſtrual Parallax, (namely, that which is made when the Moon appears preciſely biſected), or the Angle Z E O, and together with this Angle the Moon's Diſtance Z O. We find the Diſtance of the Sun by this Analogy. As the Sine of the Angle Z E O is to the Sine of the Angle E O Z; which Sine is the Radius: So is Z O, the Moon's Diſtance, unto Z E the Diſtance of the Sun.* Q. E. I.

PROP.

PROP. XXI. Theorem.

THE *Angles (B Q C, B F C) which in a Circle ſtand upon the ſame Arch (B O C); or which are in the ſame Segment (B Q S C), are all equal among themſelves.* Fig. 33.

Let firſt the Segment B Q S C be greater than a Semicircle, From the Centre A draw AB, AC. By the foregoing the Angle B A C at the Centre is double of each B Q C, B F C. Therefore they all B Q C, B F C are equal[a]. *Q. E. D.*

Then let the Segment B Q B be equal to or leſs than a Semicircle. In the Triangles B Q I, C F I, becauſe the Angles vertically oppoſite at I are equal[b], the Sum of the reſt, Q and R will be equal to the Sum of the[c] reſt F and O. Wherefore if from theſe equal Sums there be taken away the Angles R and O, which by the firſt Part are equal, as ſtanding upon the ſame Arch Q F, the Angles which remain Q, F muſt be equal. *Q. E. D.* Fig. 34.

Coroll. *Hence we gather in Opticks, that any Line B C to the Eye placed where you will in the Circumference of the Circle, whereof the Line is a Chord, appears of the ſame Magnitude; to wit, becauſe it appears every where under an equal Angle B Q C.*

[Schol. *If, of two equal Angles ſtanding upon the ſame Arch, one of them be at the Circumference, the other alſo will be at the Circumference.*

[a] *Per Axiom* 6.

[b] *Per* 15. *l.* 1.

[c] *Per Corol.* 10. *p.* 32. *l.* 1.

Fig. 33, 34. *If it be denied, B Q C ſhall either be equal to the Angle B I C on this Side the Circumference Q F, or to the Angle B E C, which is beyond the ſaid Circumference. But the Angle B I C is*[d] *greater, and the Angle B E C*[d] *is leſs than the Angle B Q C. Therefore,* &c.]

[d] *Per Corol.* 1. *p.* 32. *l.* 1.

PROP. XXII. Theorem.

Fig. 35. *IN any Quadrilateral inſcribed in a Circle (A B C F) the oppoſite Angles make two right ones.*

Let B F, C A be drawn. The Angle ABC with the[e] two O and X make two right Angles. But O is equal to I[f], becauſe it ſtands upon the ſame Arch B C: And again, X is equal to Z[f], becauſe it ſtands upon the ſame Arch AB. Therefore ABC taken together with the two Angles I and Z, that is, with the whole oppoſite Angle AFC, makes two right Angles. *Q. E. D.*

[e] *Per* 32. *l.* 1.

[f] *Per* 21. *l.* 3.

[Corollary, (1.) *Hence if one Side of a Quadrilateral deſcribed in a Circle be protracted, the external Angle will be equal to the oppoſite Angle of the Quadrilateral; for the internal added to either of them makes two right Angles.*

(2.) *Likewiſe a Circle cannot be deſcribed about a Rhombus; becauſe its oppoſite Angles either fall ſhort of, or exceed two right Angles.*

(3.) *Likewiſe if in any Quadrilateral ABCF the oppoſite Angles F and B are equal to two right ones, a Circle may be deſcribed about it. For*[g] *a Circle will paſs through any three Angles C, F, A, and this ſo that the*[h] *fourth be equal to B; which cannot*

[g] *Per* 5. *l.* 4.

[h] *Per* 22. *l.* 3.

cannot be unleſs it doth indeed paſs through the Point B[i]. *Therefore it doth paſs through it.*

PROP. XXIII, XXIV. Theorems.

ARE not neceſſary; and they treat of ſimilar Segments, which cannot rightly be defin'd without Proportions.

PROP. XXV. Problem.

TO perfect a given Arch (A B C). Fig. 36.

Let there be ſubtended at Random the two right Lines A B, C B; which biſect in I and L. From I and L raiſe Perpendiculars meeting one another in O. This ſhall be the Centre of that Circle whereof A B C is a Portion.

For [k] the Centre is both in the Line I X, and in the Line L Z. Therefore it is in their common Point O.

The Practice. From the Centre B taken in the Arch deſcribe a Circle: and with the ſame Interval from other Centres in the Arch deſcribe two other Circles, each of which cuts the former twice. Two right Lines drawn thro' the Interſections, and croſſing each other in O, will give the Centre.

PROP. XXVI, XXVII. Theorems.

IN equal Circles equal right Lines (C E, F I) ſubtend equal Arches; and, if the Arches are equal, the Subtenſes are alſo equal. Fig. 37.

H These

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[i] *Per Schol. pr.* 21. *l.* 3.

[k] *Per Corol. pr.* 1. *l.* 3.

Theſe two Propoſitions are plainly Axioms, and need no Demonſtration.

Fig. 55. [Coroll. (1.) *If in a Circle ABCD the Arch AB be equal to the Arch DC; AD will be parallel to BC. For, AC being drawn, the Angles ACB, CAD, as ſtanding on equal Arches, will be equal. Wherefore AD*[l] *is parallel to BC.* Q. E. D.

[l] *Per* 27. *l.* 1.

Fig. 56. (2.) *The right Line EF, which is drawn from the Point A, the middle Point of ſome Arch, and toucheth the Circle, is parallel to the right Line BC, which ſubtends that Arch. For from the Centre D draw unto the Point of Contact A the right Line DA, and join DB, DC. The Side DG is common, and DB is equal to DC, and the Angle BDA equal to the Angle CDA, the Arches BA, CA being ſuppoſed to be equal. Therefore the Angles DGB, DGC are equal*[m], *and conſequently are right Angles. But the internal Angles GAE, GAF are alſo right Angles*[n]. *Therefore BC, EF are parallel*[o]. Q. E. D.

[m] *Per* 4. *l.* 1.

[n] *Per* 18. *l.* 3.

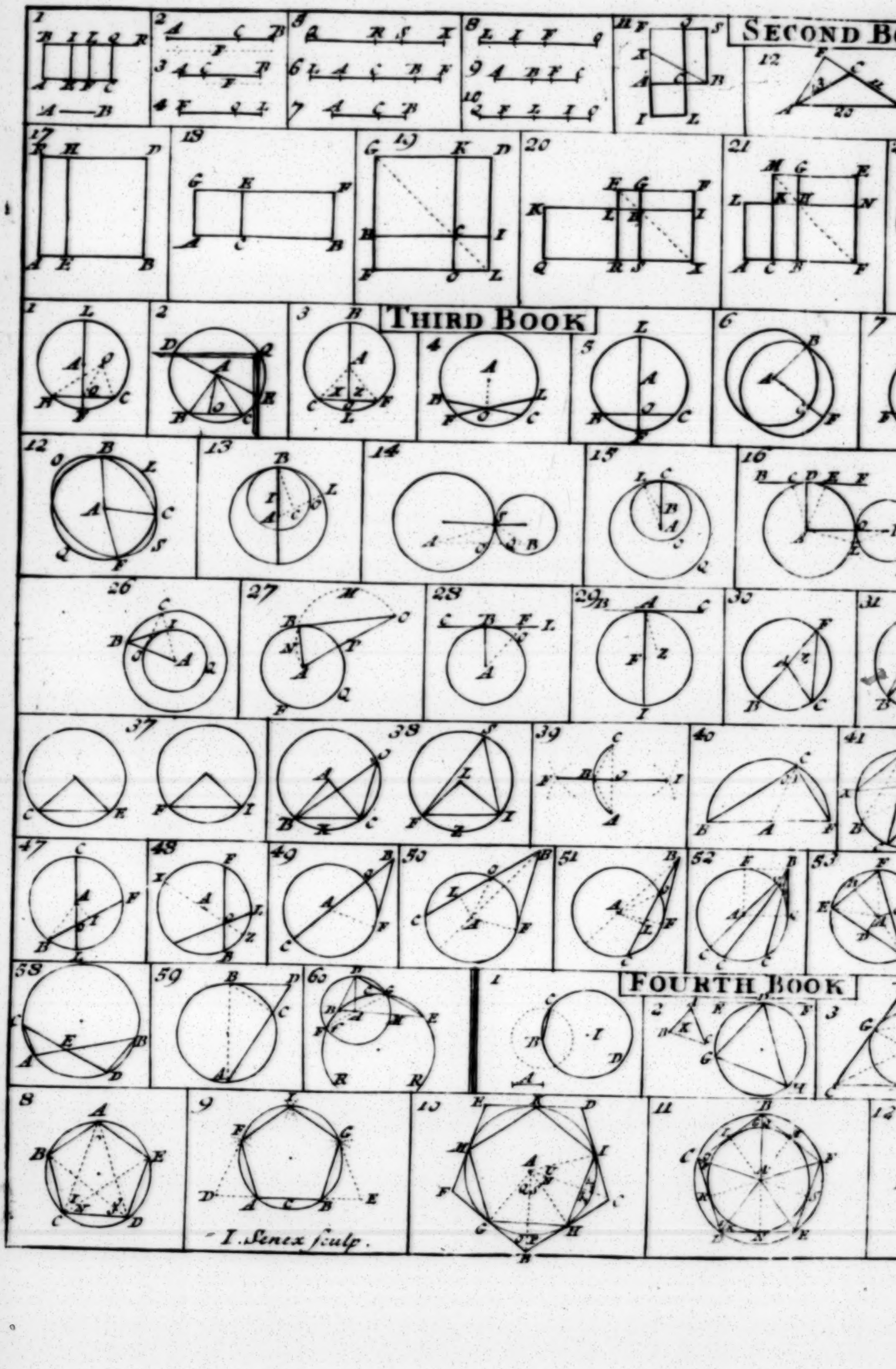
[o] *Per* 28. *l.* 1.

PROP. XXVIII, XXIX. Theorems.

Fig. 38. *IF in equal Circles the Angles whether at the Centres (BAC, FLI) or at the Circumference (BOC, FSI) be equal; the Arches alſo (BXC, FZI) on which they ſtand are equal; and, if the Arches be equal, the Angles alſo are equal.*

Theſe two Propoſitions alſo are plainly Axioms, and need no Demonſtration.

PROP.



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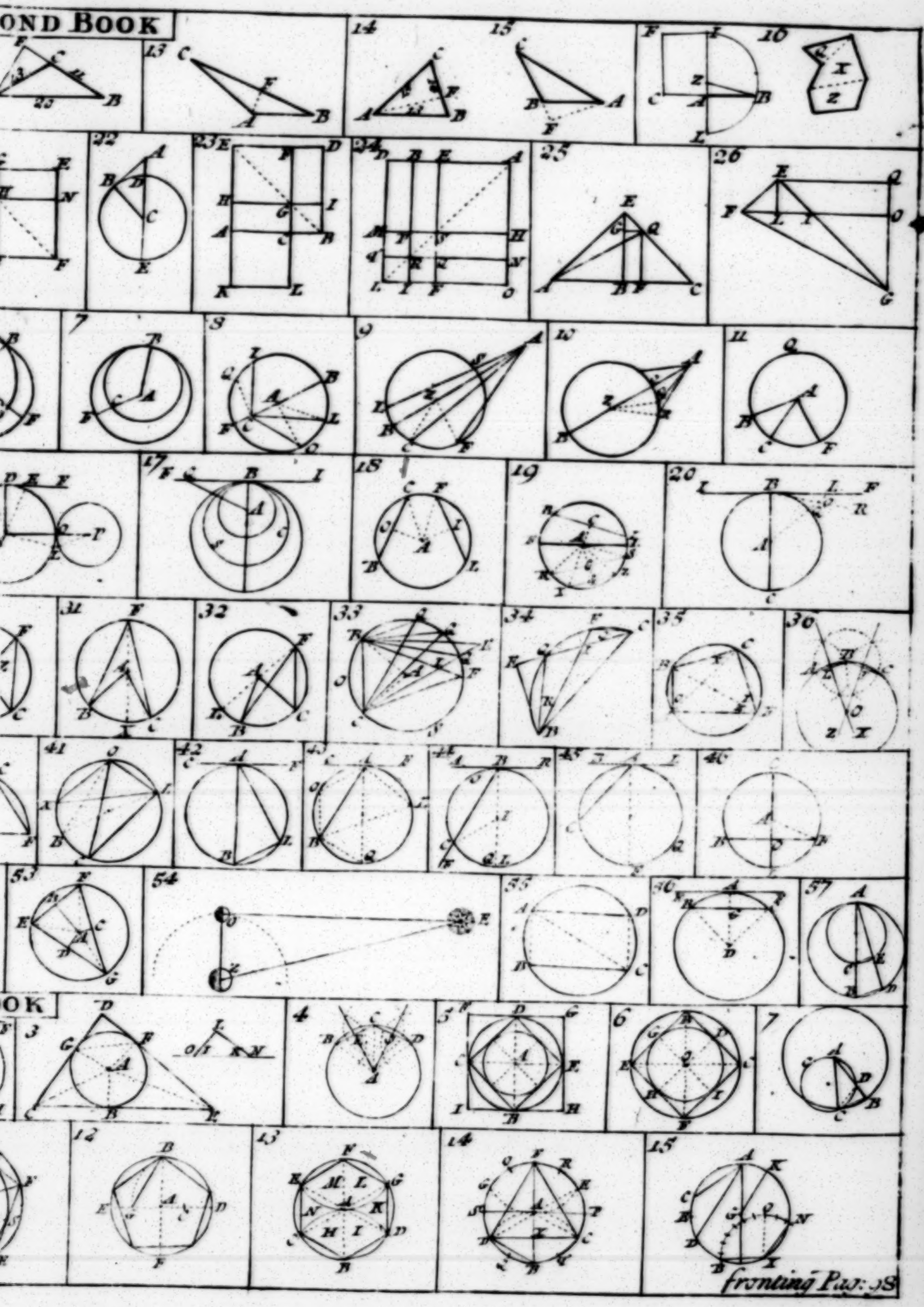

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PROP. XXX. Problem.

T*O biſect a given Arch (ABC).* Fig. 39.

Draw AC, which biſect in O. From O draw the Perpendicular OB, meeting the Arch in B. I ſay the Thing is done.

For let AB, BC be join'd. The Sides AO, OB are by the Conſtruction equal to CO, OB; and the Angles at O are equal, as being right ones. Therefore the Baſes AB, CB are equal[p]. Therefore the Arches alſo[q] AB, BC are equal.

The Practice. From the Centres A and C deſcribe with an equal Interval Arches cutting each other in the Points F and I, the right Line drawn through theſe Points will biſect the Arch ABC.

[p] *Per* 4. *l.* 1.

[q] *Per* 26. *l.* 3.

PROP. XXXI. Theorem.

T*HE Angle (BCF) in a Semicircle is a right one*; *that in a Segment greater than a Semicircle is leſs than a right one*; *that in a Segment leſs than a Semicircle is greater than a right one.* Fig. 40.

Part 1. From the Centre A draw AC. Becauſe AB and AC are equal, the Angles O and B are equal[r]. For the ſame Cauſe the Angles I and F are equal. Therefore the Angle BCF is equal to B and F together. Seeing[s] therefore the three together make two right Angles, BCF which is half of two right Angles, is one right Angle.

[r] *Per* 5. *l.* 1.

[s] *Per* 32. *l.* 1.

Fig. 41. Part 2. Let the Segment L O B F be greater than a Semicircle, and in it let there be the Angle F O L, and let LB the Diameter of the Circle be drawn. The Angle F O L is leſs than that B O L, which by Part 1. is a right one. Therefore, *&c.*

Fig. 41. Part 3. Let the Segment L O X be leſs than the Semicircle L O B, and X O L be the Angle in it. This will be greater than B O L which is a right one. Therefore, *&c.*

Fig. 40. Corollary. *Hence we may take a Proof of the Inſtrument, called a Square, whether it be exactly Rectangular or not. For in what Circle ſoever the Top of the Square is laid upon C, or any Point of the Circumference whatſoever, if the Sides of it do paſs through the Points of the Diameter B and F, the Angle is a right one; otherwiſe not.*

(2.) [*If the Sides of a Square be held continually upon the Points B and F, in the mean while that the Angle is moved round, firſt on one Side, then on the other, the Top of the Angle C will deſcribe a Circumference of a Circle, whoſe Diameter is the Line B F.*]

(3.) *Hence we learn to raiſe a Perpendicular at the End of a Line. Let B C be the Line, C the Point given, from whence a Perpendicular is to be rais'd. From any Point whatſoever A as the Centre, let a Circle be deſcribed paſſing thro' the Point C, and cutting B C in any Point, as B. If the Semidiameter B F be drawn, it is manifeſt that the Line C F is the Perpendicular required.* Q. E. F.

Fig. 57. (4.) [*Hence it is manifeſt, that Circles, touching one another inwardly, do cut all Lines, as A D proportionably; or ſo, that A E the Subtenſe of* *of the leſſer, is to AD the Subtenſe of the greater Circle, as AC, the Diameter of the leſſer, is to AB the Diameter of the greater. For there being drawn the Subtenſes EC, BD, the Triangles EAC, DAB are equiangled. For the Angle A is common, and AEC, ADB are right ones, as being Angles in a Semicircle; and therefore ACE, ABD*[t] *are equal. The Triangles therefore are ſimilar, by the fourth Propoſition of the ſixth Book, and AC : AB :: AE : AD.* Q. E. D.

Fig. 40.

(5.) *In a right-angled Triangle BCF, if the Hypotenuſe BF be biſected in A, the right Line AC cuts the Triangle into two equicrural ones ACB, ACF, and ſo a Circle deſcribed from the Centre A through B muſt paſs through C, the Top of the right Angle.*]

PROP. XXXII. Theorem.

Fig. 42, 43.

IF a right Line (CF) touch a Circle, and another (AB) which is drawn from the Point of Contact (A) cut it, the Angle, made by the Tangent and the cutting Line, is equal to the Angle which is made in the alternate or oppoſite Segment.

That is, the Angle CAB will be equal to the Angle L, which is made in the Segment ALB; and the Angle FAB will be equal to the Angle O, which is made in the Segment AOB. For,

Fig. 42.

Firſt, let the Line AB paſs through the Centre. Here by *Prop.* 18. CAB is a right Angle: and by *Prop.* 31. L is alſo a right one. Therefore CAB and L are equal.

[t] *Per Corol.* 9. *p.* 32. *l.* 1.

Fig. 43. Then let the Line A B not paſs through the Centre. Let the Line A Q therefore be drawn through the Centre, and B Q be join'd. Becauſe the Angle in the Semicircle A B Q[u] is a right one, B Q A taken together with B A Q will make one right Angle[w]. But C A Q is alſo by *Prop.* 18. of this Book a right Angle. Therefore B Q A with B A Q are equal to C A Q. The common Angle therefore B A Q being taken away, there remains B Q A, which is equal to L[x] equal to C A B. Therefore L and C A B are equal: Which is the firſt Part to be proved.

[u] *Per* 31. *l.* 3.

[w] *Per* 32. *l.* 1.

[x] *Per* 21. *l.* 3.

Then F A B and C A B make two right Angles[y], and in the Quadrilateral B O A L, the Oppoſites L and O make two right Angles[z]. Therefore the two F A B, C A B are equal to the two O and L. Therefore there being taken away on one Side C A B, on the other L, which have already been ſhew'd to be equal, there remains F A B equal to O. Which was the other Part to be proved.

[y] *Per* 13. *l.* 1.

[z] *Per* 22. *l.* 3.

PROP. XXXIII. Problem.

Fig. 44. *UPON a given Line (BC) to make a Segment of a Circle, in which the Angle ſhall be equal to any Angle given.*

Firſt let there be an acute Angle given A B F, from B draw B L perpendicular to A B; and at C, the Extremity of the Line B C, make B C I equal to C B I. (by 23. *l.* 1.) whoſe Side ſhall cut B L in I. From the Centre I deſcribe a Circle through B: This Circle will alſo paſs through C (foraſmuch as, becauſe of the Equality of the Angles at B and C, the Sides likewiſe CI, BI are (by 6. *l.* 1.) equal) and the Segment BQC ſhall contain an Angle equal to the given one ABF.

For, becauſe AB is perpendicular to the Diameter BL, AB will touch the Circle which BC cuts[a]. Therefore the Angle in the Segment BQC is equal[b] to the Angle ABF.

But, if the Angle given be obtuſe as RBC, do as before, and COB will be the Segment required.

[a] *Per* 18. *l.* 3.

[b] By the foregoing.

PROP. XXXIV. Problem.

FROM a given Circle to take away a Segment containing an Angle equal to a given one. Fig. 45.

Unto the Diameter of the Circle FA draw the Perpendicular BAL. Then[c] let AC be drawn, which may make the Angle BAC equal to that which is given. This Line AC ſhall cut off the Segment AQC, whoſe Angle is equal to the given one: As is manifeſt from *Prop.* 32.

[c] *Per* 23. *l.* 1.

PROP. XXXV. Theorem.

IF in a Circle two right Lines (CL, BF) cut one another, the Rectangle (COL) under the Segments of one is equal to the Rectangle (BOF), under the Segments of the other. For, Fig. 46, 47, 48.

If they interſect each other in A the Centre of the Circle, the Thing is manifeſt.

If one of them CL paſſeth through the Centre A, and biſects the other BF which doth not paſs through the Centre; it [d] cuts it perpendicularly, and ſo the Square of FO is the ſame with the Rectangle FOB. Let AF be drawn. Becauſe CL is biſected in A and otherwiſe divided in O,

It will be thus:

$$\left.\begin{array}{r}\text{Rect. } COL\\ +\ AOq.\end{array}\right\}\text{ will be equal to } ALq.\text{[e]}$$

$$\text{that is, to } AFq.$$

$$\text{that is, equal to }\left.\begin{array}{r}AOq.\\ +\ FOq.\end{array}\right\}\text{[f]}$$

Therefore the common Square AO being taken away, there will remain,

Rect. COL equal to FOq. that is, to the Rect. FOB.

Fig. 47. Then if one of the right Lines CL paſſes through the Centre, and cuts the other BF unequally in O, let a right Line drawn from the Centre A cut BF into two equal Parts in I. In this Caſe AIB will be a right Angle [g]. Now becauſe CL is biſected in A, and otherwiſe in O, it will be thus,

$$\left.\begin{array}{r}\text{Rect. } COL\\ +\ AOq.\end{array}\right\}\text{will be equal to } ALq\text{ [h] that is, to}$$

$$ABq.\text{ that is, to}$$

$$\left.\begin{array}{r}AIq.\\ +BIq.\end{array}\right\}\text{[i]}$$

But AOq is equal to OIq. + AIq. [k]. Therefore,

$$\left.\begin{array}{r}\text{Rect. } COL\\ +\ OIq.\\ +\ AIq.\end{array}\right\}\text{ equal to }\left.\begin{array}{r}AIq.\\ +\ BIq.\end{array}\right\}$$

Therefore

[d] *Per* 3. *l.* 3.

[e] *Per* 5. *l.* 2.

[f] *Per* 47. *l.* 1.

[g] *Per* 3. *l.* 3.

[h] *Per* 5. *l.* 2.

[i] *Per* 47. *l.* 1.

[k] By the ſame.

Therefore the common Square A I being taken away, there remains,

$$\left.\begin{matrix}\text{Rect. COL}\\ +\text{OIq.}\end{matrix}\right\} = \text{BIq.}$$

But BI Square is equal to the Rectangle Per 5.
FOB, together with OI Square: [l] becauſe l. 2.
FB is biſected in I, and otherwiſe cut in O. Therefore,

$$\left.\begin{matrix}\text{Rect. COL}\\ +\text{OIq.}\end{matrix}\right\} \text{ are equal to Rect. } \left.\begin{matrix}\text{FOB}\\ +\text{OIq.}\end{matrix}\right\}$$

Therefore the common OIq. being taken away, there remains,

$$\text{Rect. COL.} = \text{Rect. FOB.}$$

But laſtly, If neither of the Lines CL, FB Fig. 48.
paſſes through the Centre: Through their common Interſection let there be drawn the right Line XZ, which paſſes through the Centre. By what hath been juſt now demonſtrated, both the Rectangle COL, and that FOB are equal to the Rectangle ZOX. Therefore COL, FOB are equal betwixt themſelves.

[*Or the Propoſition may be demonſtrated more* Fig. 58.
eaſily and univerſally thus: Join AC and BD.
Here becauſe of the Equality of the Angles CEA,
BED as being vertically oppoſite [m]; *and of the* [m] Per 15.
Angles C and B as being upon the ſame Arch l. 1.
AD [n]; *the Triangles CEA, BED are equi-* [n] Per 21.
angled (per Corol. 9. p. 32. l. 1.) *There-* l. 3.
fore [o] $CE : EA :: EB : ED$. *Therefore* [o] Per 4.
$CE \times ED$ *is equal to* $EA \times EB$ (per 16. l. 6.) l. 6.
Q. E. D.

PROP. XXXVI. Theorem.

IF *from (B) a Point given without a Circle there be drawn unto the Circle two* Fig. 49, 50, 51.
right

right Lines, one (BF) touching it, the other (BC) cutting it; the Rectangle (CBO) which is comprehended under the whole cutting Line (CB) and the Part (BO) which lies betwixt the Point (B) and the Circle, is equal to the Square of the Tangent (BF).

1. If the cutting Line BC paſſes through the Centre A, join AF. This with the Line FB will make a right Angle [p]. And therefore becauſe CO is biſected in A, and to it is added OB; it will be thus,

$$\left.\begin{array}{r}\text{Rect. CBO}\\ +\text{AOq.}\end{array}\right\}\text{ will be equal to ABq.}^{q}\text{ that is to }\left.\begin{array}{r}\text{AFq.}\\ +\text{FBq.}\end{array}\right\}^{r}.$$

Therefore the equal Squares AOq. AFq. being taken away on both Sides, there remains,

$$\text{Rect. CBO,} = \text{BFq.}$$

[p] *Per* 18. *l.* 3.

[q] *Per* 6. *l.* 2.

[r] *Per* 47. *l.* 1.

Fig. 50, 51. 2. But then if CB doth not paſs through the Centre, let there be drawn AB, AF, AO, and AL, and let AL biſect OC in L. The Angle ALO is therefore a right one [s]. Likewiſe AFB is a right Angle [t]. Now becauſe CO is biſected in L, and to it is added OB, it will be thus,

$$\left.\begin{array}{r}\text{Rect. CBO}\\ +\text{LOq.}\end{array}\right\} = \text{LBq.}^{u}$$

Let there be added on both Sides AL Square, and then

$$\left.\begin{array}{r}\text{Rect. CBO}\\ +\text{LOq.}\\ +\text{ALq.}\end{array}\right\}\text{ equal to }\left.\begin{array}{r}\text{LBq.}\\ +\text{ALq.}\end{array}\right\}$$

But the Squares of LO, AL are equal [w] to the Square of AO, or AF; and the Squares of

[s] *Per* 3. *l.* 3.

[t] *Per* 18. *l.* 3.

[u] *Per* 6. *l.* 2.

[w] *Per* 47. *l.* 1.

of LB, AL are equal to the Square of AB[x]. [x] By the ſame.
Therefore,

$$\left.\begin{array}{l}\text{Rect. } CBO\\ +AFq.\end{array}\right\} = ABq. \text{ that is, [y] } \text{to } \left.\begin{array}{l}BFq.\\ +AFq.\end{array}\right\}$$

[y] By the ſame.

Therefore the common Square, that of AF being taken away, there remains,

Rect. CBO equal to the Square of BF. *Q. E. D.*

[*Or more eaſily and univerſally thus: Draw AB and BC. Now becauſe of the Equality of the Angles A, and DBC*[z]; *and for that the Angle D is common to both; the Triangles BDC, ADB are equiangled*[a]. *And therefore* (by 4. lib. 6.) *AD : DB :: BD : DC. Wherefore the Rectangle*[b] *AD×DC is equal to the Rectangle DB×DB, or DBq.* Q. E. D.]

Fig. 59.
[z] *Per* 32. *l.* 3.
[a] *Per Corol.* 9. *p.* 32. *l.* 1.
[b] *Per* 16. *l.* 1.

Corollaries.

1. IF from the ſame Point B without the Circle, as many cutting Lines BC as you will be drawn, all the Rectangles CBO are equal amongſt themſelves. For each of them is equal to the Square of the Tangent BE. Fig. 52.

2. The right Lines, which from the ſame Point touch the Circle, are equal. For each of their Squares is equal to the ſame Rectangle.

[3. *It is alſo clear, that, from the ſame Point B taken without the Circle, there can only two Lines BF, BQ be drawn, which ſhall touch the Circle. For, if a third be ſaid to touch it, it muſt be equal to BF, or BQ, and therefore the ſame with one of them.*

4. *In every right-angled Triangle BFA, that is not alſo an Iſoſceles, the Rectangle ariſing from the* Fig. 49.

the Sum of the Hypotenuſe, and one Side drawn into the Difference betwixt them, is equal to the Square of the other Side. For the Sum of the Hypotenuſe $BA, + AF$ *or* AC, *is* $= BC$. *And their Difference is* $BA - AF = BA - AO = BO$. *And the other Side of the Triangle is* BF. *But the Rectangle* CBO *is equal to the Square of* BF. *Therefore,* &c.]

PROP. XXXVII. Theorem.

Fig. 25. *IF the Rectangle under CB and OB be equal to the Square of BF, this muſt touch the Circle in F.*

From B let there be drawn the Tangent BQ, and the right Lines EQ, EF being drawn from the Centre E, unto the Points Q and F, let BE be joined. Becauſe by the Suppoſition the Square of BF is equal to the Rectangle CBO, as is alſo the Square of BQ, by 36. of this Book, the Squares of BQ, BF ſhall be equal betwixt themſelves, and conſequently the right Lines BQ, BF are equal. Therefore the Triangles FEB, BEQ are equilateral each to other. Therefore the Angles Q, F are equal [c]. But Q is a right Angle (*per* 18. *l.* 3.) Therefore F alſo is a right Angle. Therefore BF toucheth the Circle [d].

[c] *Per* 8. *l.* 1.

[d] *Per* 16. *l.* 3.

[Corollaries 1. *Hence the Angle EBF is equal to the Angle EBQ* (per 8. l. 1.)

(2.) *If two equal right Lines BF, BQ fall from ſome Point B upon the convex Circumference, and BF one of them toucheth the Circle, the other BQ muſt touch it alſo. For ſeeing BF, BQ*

B Q are equal, their Squares are alſo equal. But B F q is equal to C B O[e]*. Therefore B Q q = C B O*[f]*. Therefore B Q alſo toucheth the Circle*[g]*.*

[e] By the foregoing.
[f] *Per. Axiom* 1.
[g] By this Propoſition.

Scholium [1.] *Seeing all Planes paſſing through the Centre of the Earth, which all ſtand perpendicular upon the Horizon, do produce great and equal Circles upon the Earth's Surface, we ſhall here bring in ſome elegant Conſectaries from thence, out of our Author in his Aſtronomy; which from the Nature of Circles may very eaſily be underſtood.*

(1.) *If any Part of the Surface of the Earth were perfectly plain, Men could no more ſtand upright upon it, than upon the Side of a Hill, ſaving in the Point of Contact only.*

(2.) *The Head of a Traveller performs a longer Way or Courſe than his Feet; likewiſe he that is on Horſeback, and goes the ſame Way as a Footman, meaſures a greater or longer Space than he that is on Foot. As likewiſe in a Ship, the uppermoſt Part of the Maſt runs over more Way than the lower Part of it.*

(3.) *If any one ſhould travel over the whole Circumference of the Earth, the Way gone over by his Head would exceed that which is gone over by his Feet, by the Difference of Circumferences; or by the Circumference of a Circle, whoſe Semidiameter is the Man's own Stature.*

(4.) *If a Veſſel full of Water be elevated perpendicularly, the Water will continually be running over, and yet it will remain full; namely, becauſe the Surface of the Water is continually compreſſed into the Surface of a greater Sphere. Yea, if a Veſſel be elevated continually higher and higher, the Surface of the Water, which is contain'd* *tain'd in it, will continually deſcend and come nearer unto a Plane; unto which yet it will never actually come.*

(5.) *If a Veſſel full of Water be carried directly downwards, although nothing run over, yet it will ceaſe to be full; namely, becauſe the Surface of the Water ſwells continually into a Part of a leſſer Sphere. From whence it follows,*

(6.) *That one and the ſame Veſſel contains more Water at the Foot of a Mountain than at the Top; as likewiſe more in a ſubterraneous Cellar, than in a Chamber. To which Things add,*

(7.) *That two Strings on which two iron Balls hang perpendicular, [and conſequently the Walls of Buildings erected perpendicularly] are not parallel one to another, but Parts of Radius's meeting together in the Centre of the Earth.*

Fig. 60. Scholium [2.] *I think it not amiſs to inſert in this Place this following Problem alſo, which was communicated to me by a Friend, as demonſtrated by me ſomewhat more briefly.*

Through the two Points (B) and (C) in a given Circle (B D M) to draw the Circumference of a Circle which ſhall biſect the Circumference of the other given Circle.

Through the Centre A and one of the given Points B let there be drawn the infinite right Line B A M E. Unto which from the Centre let there be erected the Perpendicular A D, and let B D be drawn. Let the Line D E be made perpendicular to B D, cutting the infinite Line B A M E in the Point E. Laſtly, let a Circle be drawn [h] *through the three Points, B, C, E. I ſay the Thing is done. For,*

[h] *Per.* 5. *l.* 4.

Let a Chord of the ſecond Circle be drawn through either of the Interſections of the Circles, as *as G, and through A the Centre of the firſt Circle, to wit,* GAf; *let alſo the Diameter of the firſt Circle* GAF *be drawn. Then in the firſt Circle* (by Corol. 1. Prop. 8. l. 6. *and by* Prop. 17. l. 6.) $AB \times AE = ADq$, *that is,* (*becauſe of the Equality of the Semidiameters,* AD, AG, AF) $= AG \times AF$. *And in the ſecond Circle there will be* [i] $AB \times AE = AG \times Af$. *Therefore* $AF = Af$, *and the Points F, f coincide, and the Arch* FDG *is equal to the Arch* FMG. Q. E. F.

[i] *Per* 35. *l.* 3.

THE ELEMENTS OF EUCLID.

BOOK IV.

THIS Book, which is wholly Problematical, teacheth by what Artifice, Figures, thoſe which are ordinate or regular eſpecially, may be inſcribed in, and circumſcribed about, Circles. There is very great Uſe of it in building Fortifications; and from it as a Fountain have been derived thoſe moſt excellent Tables of Sines, Tangents, and Secants, to the very great Benefit of the Mathematicks.

[*This Book is moſt uſeful for* Trigonometry. *For, by inſcribing Polygons in a Circle, we learn to frame Tables of Chords, Tangents, and Secants: by the Help of which we learn to meaſure the Magnitudes of Figures and Bodies. Neither without it can we duly diſtinguiſh the Aſpects, as they call them,*

them, of the Stars, as the Quartile, Sextile, &c. they wholly depending upon the Inſcription of Polygons in a Circle. Neither can we otherwiſe collect the Area (which is a certain Quadrature of a Circle) than from the Area's or Squares of innumerable Polygons inſcrib'd in, and circumſcrib'd about, a Circle. And in like manner we know the duplicate Proportion of Circles amongſt themſelves, from the duplicate Proportion of Polygons inſcrib'd in, or circumſcrib'd about, Circles. And as for military Architecture, it makes ſo much Uſe of Polygons inſcrib'd in Circles, that more than all other Sciences it may ſeem to be wholly owing to this Book.]

DEFINITIONS.

1. A Rectilineal Figure is ſaid to be inſcrib'd in a Circle, or to a have Circle circumſcrib'd about it, when the Tops of all the Angles thereof are in the Circumference of the Circle.

2. A rectilineal Figure is ſaid to be circumſcrib'd about a Circle, or to have a Circle inſcrib'd in it, when each one of its Sides toucheth the Circle.

3. An ordinate or regular Figure is that which is equilateral and equiangular.

PROPOSITION I. Problem.

TO inſcribe a right Line (A) which is not greater than the Diameter into a Circle (B D). Fig. 1. *l.* 4.

Take in the Circumference any Point B. From the Centre B with the Interval of the given

ven Line A deſcribe an Arch, cutting the Circle in C. Draw the right Line BC. I ſay the Thing is done.

PROP. II. Problem.

Fig. 2.

TO inſcribe in a Circle a Triangle having equal Angles with a given one (X).

Let the Line EF touch the Circle in D. Let EDG be made [a] equal to the Angle C, and FDH equal to B; and join GH. I ſay the Thing is done. For [b] EDG is equal to H. H conſequently is equal to the Angle C [c]. And FDH is equal to [d] G; and conſequently G to B. Therefore GDH [e] is equal to the Angle A. Therefore what was required is done.

[a] *Per* 23. *l.* 1.
[b] *Per* 32. *l.* 3.
[c] By the Conſtruction.
[d] *Per* 32. *l.* 3.
[e] *Per Corol.* 9. *p.* 32. *l.* 1.

PROP. III. Problem.

Fig. 3.

TO circumſcribe about a Circle a Triangle, having equal Angles with a given one (ILK).

Let the Line IK be drawn forth on both Sides, ſo as to make the external Angles O and N. Make, at the Centre A, the Angles GAB, BAF equal to O, N reſpectively, which is done by 23. *l.* 1. Then in the Points G, F, B, let three right Lines touch the Circle, meeting together in C, E, D. The Triangle CED is circumſcrib'd about the Circle, and is equi-angled to the given one ILK. For,

In the Quadrilateral CGAB, the Angles G and B are [f] both of them right ones. Therefore

[f] *Per* 18. *l.* 3.

fore the remaining ones G A B and C taken together do[g] make two right ones, and conſequently are equal to the two together, O, I. Therefore the two G A B and O, which are equal by the Conſtruction, being taken away, there remains C equal to I. In the ſame manner E will be prov'd equal to the Angle K. Therefore D and L are[h] likewiſe equal. That therefore is done which was demanded.

For that the Tangents do concur is thus ſhew'd: The Angles O, I, and K, N are[i] equal to four right ones; and I, K are leſs than two right ones[k]. Therefore O, N, (that is, by the Conſtruction G A B and B A F) are greater than two right ones. Therefore G A F[l] is leſs than two right ones. Therefore G F falls between A and D. Therefore ſeeing A G D and A F D are right Angles, D G F and D F G are leſs than two right ones. Therefore C G D and E F D[m] meet together towards D. In the ſame manner it may be demonſtrated that the reſt concur.

PROP. IV. Problem.

To inſcribe a Circle in a Triangle.

Biſect the two Angles C and E with the Lines C A, E A, meeting together in A. From A draw the Perpendiculars, A B, A G, A F. A Circle, deſcribed from the Centre A through B, will paſs alſo through G and F, and touch the three Sides of the Triangle. Fig. 3

For in the Triangles C A G, C A B, becauſe the Angles A G C, A B C, and likewiſe thoſe G C A, and B C A are equal by the Conſtruction,

[g] *Per Theo.* 1. *Schol. p.* 32. *l.* 1.

[h] *Per Corol.* 9. *p.* 32. *l.* 1.

[i] *Per* 13. *l.* 1.

[k] *Per* 32. *l.* 1.

[l] *Per Corol.* 3. *p.* 13. *l.* 1.

[m] *Per Schol. p.* 31. *l.* 1.

ſtruction, and the Side A C is common, the Sides A G, A B[n] muſt be likewiſe equal. In like manner A B, A F may be ſhewn to be equal. Therefore the Circle, deſcrib'd from the Centre A, paſſeth through B, G, F. And, becauſe the Angles at thoſe three Points are equal, it toucheth[o] all the Sides of the Triangle. That therefore is done which was required.

[n] *Per* 26. *l.* 1.

[o] *Per* 16. *l.* 3.

[*Hence the Sides of a Triangle being known, the Segments of them which are made from the Contacts of an inſcribed Circle will be known. Let* DC *be* 12. DE 18. CE 16. DC *and* CE *will be* 28. *from which ſubtract* $18 = DE = DG + BE$, *there remains* $10 = CG + CB$. *Therefore* CG *or* $CB = 5$. *Conſequently* EB *or* $EF = 11$. *Wherefore* FD *or* $DG = 7$.]

PROP. V. Problem.

Fig. 4.

TO deſcribe a Circle about a Triangle, or through three given Points B, C, D, not lying in a right Line, to deſcribe a Circle.

Connect the given Points with two right Lines B C, C D, which biſect with the Perpendiculars E A, O A, meeting together in A. This will be the Centre of a Circle which paſſeth through B, C, D.

Let the right Lines A C, A D, A B be drawn. By the Conſtruction the Sides D O, O A, are equal to theſe C O, O A; and the Angles at O are right ones. Therefore A D is equal to A C[p]. In the ſame manner A B may be prov'd equal to A C. Therefore A D, A B[q] are equal. Therefore a Circle, deſcribed from

[p] *Per* 41. *l.* 1.

[q] *Per Axiom* 1.

from the Centre A through B, will paſs alſo through C and D. Which was the Thing required.

As for the Practice, it is ſufficient to deſcribe from B, C, D three equal Circles, interſecting each other; and through the Interſections to draw right Lines, theſe meeting one another will give the Centre ſought.

PROP. VI, VII. Problems.

TO inſcribe a Square in, and circumſcribe one about a Circle. Fig 5

Let the Diameters BD, CE be drawn, cutting each other perpendicularly. The right Lines, which join the Terms of theſe, inſcribe a Square in a Circle.

The Demonſtration is manifeſt from 4. *l.* 1. and 31. *l.* 3. Then let four Tangents be drawn touching the Circle in B, C, D, E, meeting together in I, F, G, H. The Figure IFGH is a Square, circumſcrib'd about a Circle.

The Demonſtration is manifeſt from 18. *l.* 3. with *Corol.* 2. *Prop.* 36. *l.* 3. and 28, and 34. *l.* 1.

Scholium.

A Square deſcrib'd about a Circle is double to that inſcrib'd. For, becauſe the Angle BCD in the Semicircle [r] is a right one, the Square of BD (that is, FI Square) ſhall be [s] equal to BCq + CDq, and therefore double to the Square of CD, *i. e.* to CDEB. Fig. 5.

[r] *Per* 31. *l.* 3.

[s] *Per* 47. *l.* 1.

PROP. VIII, IX. Problems.

Fig. 6. *TO inſcribe a Circle in, and circumſcribe one about a Square, (as B C F E).*

Let there be drawn the Diameters of the Square, cutting each other in A. From the Centre A deſcribe a Circle through B; this will alſo paſs through E, F, C.

Then from the Centre A draw A D perpendicular to B C; a Circle, deſcrib'd from the Center A through D, will touch all the Sides of the Square.

Part I. Becauſe by the Hypotheſis the Lines CB, EB are equal; the Angles B C E, B E C will be equal[t]. But CBE is a right Angle by the Hypotheſis. BCE therefore and BEC are half right ones[u]. In the ſame manner CBF will be ſhew'd to be an half right Angle, as likewiſe the reſt of the Angles; and ſo they are equal amongſt themſelves. Therefore in the Triangle B A C, ſeeing there are two equal Angles CBA, BCA, the right Lines AB and AC[w] are equal. In the like manner the right Lines A B, A E, A F may be ſhew'd to be equal. Therefore a Circle, deſcribed from the Centre A through B, paſſes through E, F, C.

Part II. From A let there be alſo drawn the Perpendiculars A G, A H, A I. Becauſe in the Triangles G B A, D B A, the Angles at D and G, as likewiſe thoſe at B are equal, and the Side A B is common, the Sides A D, A G muſt be equal[x]. In the ſame manner A G, A H, A I may be ſhew'd to be equal. Therefore

[t] *Per* 5. *l.* 1.

[u] *Per Corol.* 11. *p.* 32. *l.* 1.

[w] *Per* 6. *l.* 1.

[x] *Per* 26. *l.* 1.

fore a Circle deſcrib'd from the Centre A, which paſſeth through D, will alſo paſs through G, H, I, and touch all the Sides of the Square[y]. Becauſe the Angles at D, G, H, I are right ones. Therefore we have done what was required.

PROP. X. Problem.

TO make an Iſoſceles Triangle BAC, in which the Angle at the Baſe (ABC, or ACB) ſhall be double to that which is at the Top (A). Fig. 7.

Let any right Line, what you will, as AB, be taken, which ſo cut in D[z] that the Rectangle ABD ſhall be equal to AD Square. Then from the Centre A through B deſcribe a Circle; in which inſcribe[a] BC equal to AD, and join AC. BAC ſhall be the Triangle ſought.

For let the right Line DC be drawn, and through A, D, C deſcribe[b] a Circle. Becauſe the Rectangle ABD is equal to the Square AD, (that is, BC) it is manifeſt, that BC[c] toucheth that Circle DO which CD cuts. Therefore the Angle BCD[d] is equal to the Angle A in the oppoſite Segment, and ſo the common Angle DCA being added, BCA muſt be equal to A + DCA. But, becauſe the Sides AB, AC are equal, ABC[e] is equal to the Angle ACB. Therefore the Angle ABC is alſo equal to A + DCA. But the external Angle alſo BDC is equal to the two internal ones[f] A + DCA. Therefore ABC and BDC are equal. Therefore the Line DC is[g] equal to BC, (that is, by the Conſtruction

[y] *Per* 16. *l.* 3.

[z] *Per* 11. *l.* 2.

[a] *Per* 1. *l.* 4.

[b] *Per* 5. *l.* 4.

[c] *Per* 37. *l.* 3.

[d] *Per* 32. *l.* 3.

[e] *Per* 5. *l.* 1.

[f] *Per* 32. *l.* 1.

[g] *Per* 6. *l.* 1.

to DA). Therefore the Angles A and DCA[h] are equal. Wherefore the Angle ABC, which hath been ſhew'd equal to thoſe two, ſhall be double to one A. That is done therefore which was required.

Corollary.

EACH of the Angles at the Baſe B and C, in the *Iſoſceles* now framed, is two Fifths of two right ones, or four Fifths of one right one, and the remaining one A is one Fifth of two right ones, or two Fifths of one right one. As is manifeſt out of this Propoſition taken together with that 32. *l.* 1.

PROP. XI. Problem.

Fig. 7, 8. *TO inſcribe a regular Pentagon in a Circle.*

Let there be deſcribed[i] the Triangle BAC, having the Angles at the Baſe double to that at the Top. Inſcribe a Triangle CAD equiangled to this in a Circle[k]. Biſect the Angles at the Baſe ACD, ADC, with the right Lines CE, DB, cutting the Circle in E and B. The Points A, B, C, D, E, join'd by right Lines, will give an ordinate Pentagon inſcrib'd in a Circle.

For from the Conſtruction it appears, that the Angles I, N, Q, S, O are equal. Wherefore the Arches ſubtended to them AE, ED, CD, CB, BA are alſo[l] equal. Therefore the right Lines ſubtended to thoſe Arches ſhall alſo[m] be equal. The Pentagon therefore is equilateral. But it is alſo[n] equiangular, becauſe its Angles BAE, AED, *&c.* ſtand on equal Arches BCDE, ABCD, *&c.* That therefore is done which was required.

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[h] *Per* 5. *l.* 1.

[i] By the foregoing.

[k] *Per* 2. *l.* 4.

[l] *Per* 28. *l.* 3.

[m] *Per* 27. *l.* 3.

[n] *Per* 29. *l.* 3.

Corollary.

Fig. 8.

THE Angle of a regular Pentagon makes ſix Fifths of one right Angle, or three Fifths of two. For the three Angles at A ſeeing they are equal, as ſtanding upon equal Arches, BC, CD, DE, and the middlemoſt of them by the Corollary foregoing is two Fifths of one right Angle; the three together, that is, the Angle of the Pentagon itſelf muſt make ſix Fifths of one right one.

Fig. 8.

[Scholium. *This holds univerſally, that Figures of an odd Number of Sides are inſcribed in a Circle, by means of an* Iſoſceles *Triangle, whoſe equal Angles at the Baſe are multiple of thoſe at the Top. But Figures of an even Number of Sides are inſcribed by the means of* Iſoſceles *Triangles, whoſe Angles at the Baſe are each of them multiple ſeſquialteral of that which is at the Top.*

As in the Iſoſceles *ACD, if the Angle C or D be threefold of A, the Side C D will be the Side of an Heptagon; if fourfold, it will be the Side of an Enneagon,* &c. *But, if C or D ſhall be* 1½ *of A, C D will be the Side of a Square; and, if C ſhall be* 2½ *of the Angle A, C D will ſubtend a ſixth Part of the Circumference: In like manner, if C or D ſhall be* 3½ *of the Angle A, C D ſhall be the Side of an Octagon,* &c.]

Scholium.

EUCLID's Inſcription of a Pentagon is ingenious, but that of *Ptolomy*, which he delivers in the firſt Book of his *Almageſt*, is much more expeditious: And it is this:

Fig. 12. Let the Diameters E D, B F be drawn, cutting one perpendicularly in A. Biſect the Radius A D in C. From the Centre C through B deſcribe an Arch, meeting the Diameter E D in G. The right Line G B is the Side of a Pentagon, and A G of a Decagon.

The Demonſtration cannot be given here, for it depends upon the 13th Book of *Euclid.* See it in *Clavius*, in his Scholium, after *Prop.* 10. *l.* 13.

Problem.

Fig. 9. UPON a given right Line (A B) to deſcribe a regular Pentagon.

° *Per* 11. *l.* 2. Cut A B ſo in C ° that the Rectangle A B C may be equal to the Square of A D. From A B protracted on both Sides take away A D, B E, equal to the greater Segment A C. From the Centres A and D with the Interval A B deſcribe two Arches, cutting each other in F. Likewiſe from the Centres B and E deſcribe, with the ſame Interval, two Arches cutting each other in G. And again, from the Centres G and F, with the ſame Interval, deſcribe two others, cutting each other in I. The Points A, F, I, G, B, being join'd, will give a regular Polygon upon the given right Line A B.

That it is equilateral is manifeſt from the Conſtruction; that it is equi-angled will be thus demonſtrated: Let DF be drawn. It is manifeſt by the Conſtruction, that ADF is an *Iſoſceles*. And the Baſe AD is the greater Segment of the Side DF, ſo divided, that the Rectangle of the Whole and the leſſer Side is equal to the Square of the greater. (For DF is equal to AB, and AD equal to AC). Therefore the Angle DAF is two Fifths of two right ones; by *Coroll. Prop.* 10. *l.* 4. Therefore the remaining Angle FAB is three Fifths of two right ones, or ſix Fifths of one right one [p]; and therefore is an Angle of a regular Pentagon [q]. In the ſame manner may it be ſhewn that the Angle GBA is three Fifths of two right ones, and ſo equal to FAB. From whence it is neceſſary, that the reſt F, G, I, ſhould be equal to theſe, as appears from their being equilateral to theſe, if the right Line FG be conceiv'd to be ſubtended.

[p] *Per* 13. *l.* 1.

[q] *Per Corol. pr.* 11. *l.* 4.

PROP. XII. Problem.

TO *circumſcribe an ordinate Pentagon about a Circle.*

Fig. 10.

Let there, by the foregoing, be inſcrib'd the regular Pentagon GHIKM, and let there be drawn Tangents in the Points G, H, I, K, M, which may concur in B, C, D, E, F. I ſay the thing is done.

For from the Centre A draw the right Lines, AG, AB, AH, AC, AI. Here, becauſe from the ſame Point B, BG and BH touch the Circle, they [r] are equal. Therefore the Triangles GAB, BAH are equilateral to each other. Therefore [s] the Angles O, P, as likewiſe

[r] *Per Corol.* 2. *p.* 36. *l.* 3.

[s] *Per* 8. *l.* 1.

likewiſe thoſe Q, S, are equal. And therefore the whole Angle B is double to P, and the whole G A H double to S. For the ſame Reaſon the Angles C and H A I are double to T and N reſpectively. But G A H and H A I are equal [s], becauſe they ſtand upon equal Arches, by Conſtruction G H, H I. Therefore their Halves S and N are alſo equal. Becauſe therefore in the Triangles B A H, H A C, the two Angles S and N are equal, and thoſe at H are both right Angles [t], and likewiſe the Side A H is common; therefore the Sides [u] B H, C H, as likewiſe the Angles P, T, are equal. In the ſame manner I might ſhew B G, F G, to be equal. Therefore B F, C B, which are double to the Equals B G, B H, are alſo equal. In the ſame manner it may be ſhew'd that the reſt of the Sides of the circumſcribed Pentagon are equal. It is therefore equilateral; but it is alſo equiangled; for, ſeeing it hath been ſhew'd that the Angles B and C are each of them double to the Equals P and T, they muſt alſo be equal betwixt themſelves. And in the ſame manner of the reſt. We have therefore deſcribed a regular Pentagon about a Circle. Which was the thing to be done.

[s] *Per* 29. *l.* 3.

[t] *Per* 18. *l.* 3.

[u] *Per* 26. *l.* 1.

In the ſame Way any ordinate Figure whatſoever is deſcrib'd about a Circle, that is, if a like Figure be firſt inſcrib'd in the Circle.

PROP. XIII, XIV. Problems.

Fig. 11.

TO inſcribe a Circle in a regular Pentagon, and circumſcribe one about it.

Biſect the two Angles of the Pentagon B, C, with the right Lines B N, C S, cutting each other in A. From A draw the Perpendicular A L.

A Circle deſcrib'd from the Point A with the Interval A L touches all the Sides of the Pentagon; and a Circle deſcrib'd from the ſame Point A with the Interval A B paſſes alſo through the Points F, E, D, C.

Part I. In the Triangles D C A, B C A, becauſe the Sides D C, C A, are equal to B C, C A, by the Hypotheſis, and the Angles P and O are equal by the Conſtruction, thoſe alſo G and I will be equal by 4. l. 1. Now the Whole alſo B and D are equal by the Hypotheſis. Wherefore ſeeing the Angle G is half of B by the Conſtruction; I alſo will be half of D. Therefore D is biſected by the right Line D M. For the ſame Cauſe the reſt of the Angles of the Pentagon E, F, are biſected, and conſequently all the half Angles are equal betwixt themſelves. Now let the Perpendiculars be drawn, A M, A S, A N, A R. For becauſe in the Triangles L B A, M B A, the Angles G and B L A are equal to the Angles Q and B M A, by the Conſtruction, and the Side B A is common, A L and A M muſt be alſo equal[w]. In like manner I might ſhew that the reſt of the Perpendiculars, A M, A N, A S, A R, are equal. A Circle therefore from the Centre A, paſſing through L, will likewiſe paſs through M, S, N, R, and, becauſe the Angles at L, M, S, N, R, are right ones by the Conſtruction[x], it will touch the five Sides of the Pentagon. Which was the firſt Part.

[w] *Per* 26. *l.* 1.

[x] *Per* 16. *l.* 3.

Part II. In the Triangle C A B, becauſe the Angles O and G have already been ſhewn to be equal, the Sides alſo A C, A B muſt be equal[y], and in the ſame manner, A B, A F, A E, A D,

[y] *Per* 6. *l.* 1.

may be prov'd equal, and therefore a Circle from the Centre A paſſing through B muſt paſs alſo through C, D, E, F. Therefore we have both inſcrib'd a Pentagon in a Circle, and circumſcrib'd one about a Circle. *Q. E. F.*

[*In the ſame Way, in any regular Figure whatſoever, a Circle may be inſcrib'd, and circumſcrib'd about it.*]

PROP. XV. Problem.

Fig. 13. *IN a given Circle to deſcribe a regular Hexagon.*

Let the Diameter FAB be drawn. From the Centre B through A deſcribe a Circle, cutting the given one in C and D. Likewiſe from the Centre F through A a Circle cutting the given one in E and G. The ſix Points, B, C, E, F, G, D, connected by right Lines, will give the Hexagon required.

From the Centre let fall the right Lines AE, AC, AG, AD. It is manifeſt that the Triangles H, I, M, L are equilateral, both in themſelves, and with one another [z]. Then becauſe the Angles CAB, EAF, each of them make one Third of two right Angles (*per Corol.* 12. *p.* 32. *l.* 1.) and therefore do make both together two Thirds of two right Angles; it remains [a] that EAC is one Third of two right Angles; therefore the Angles EAC, CAB are equal. But the Sides alſo EA, AC, are equal to the Sides BA, AC. Therefore the Baſe EC (*per* 4. *l.* 1.) is equal to the Baſe BC, that is, to the Radius AC by the Conſtruction. Wherefore the Triangle N is alſo equilateral. And in the ſame manner the Triangle

[z] *Per* 1. *l.* 1.

[a] *Per Corol.* 1. *p.* 13. *l.* 1.

angle K may be ſhewn to be ſo. Becauſe therefore all the ſix Triangles, H, I, K, L, M, N are equilateral; it is manifeſt that all the Sides, CB, BD, DG, GF, FE, EC, are equal one to another, and to the Radius AC. The Hexagon is therefore equilateral. But it is alſo equiangular, ſeeing each one of its Angles E, C, B, D, G, F, conſiſts of two Angles of an equilateral Triangle. Therefore we have inſcribed a regular Hexagon in the Circle.

Corollaries.

1. THE Side of an Hexagon inſcrib'd in a Circle, [or a Chord of 60 Degrees] is equal to the Radius [and conſequently the Sine of 30 Degrees is equal to half the Radius (*per Corol.* 2. *p.* 3. *l.* 3.)]

2. An Angle of a regular Hexagon is four Thirds of one right Angle; as conſiſting of two Angles of an equilateral Triangle, each of which makes two Thirds of a right Angle.

3. If there be drawn the Diameter PS, perpendicular to the other FB; and with the Interval of the Radius PA, from the Centre P, and S, there be made Sections in O and Q, in R and T, the Points, P, E, O, F, R, G, S, D, T, B, Q, C, connected with right Lines, will give a Figure of 12 Sides, inſcrib'd in a Circle with one Aperture of the Compaſſes. Which Thing is of great Service in Dialling. Fig. 14.

4. From what has been demonſtrated we may eaſily deſcribe an equilateral Triangle in a Circle. The Diameter FB being drawn, from the Centre B through A deſcribe the Arch CAD. The Points C, F, D, connected Fig. 14.

ed with right Lines, will give the Triangle ſought.

5. The Side CXD of the equilateral Triangle cuts off from the Diameter BF, perpendicular to it, a fourth Part thereof BX. For the Angles ACX, BCX, ſtanding upon equal Arches GD, DB, are equal (*per* 29. *l.* 3.) and the Sides AC, CX, are equal to the Sides BC, CX. Therefore AX, BX are equal[b]. Therefore BX is the fourth Part of the Diameter BF.

[b] *Per* 4. *l.* 1.

Scholium 1. *Problem.*

Fig. 13.

YOU may raiſe a regular Hexagon upon a right Line BC thus: Make an[c] equilateral Triangle CAB upon the given Line CB. From the Centre A through B and C deſcribe a Circle. This will contain an Hexagon upon the given right Line CB. The Thing is manifeſt from the Propoſition, and *Coroll.* 1.

[c] *Per* †. *l.* 1.

Theorem.

THE Square of a Side of an equilateral Triangle is treble to the Square of the Semidiameter of a Circle in which it is inſcrib'd, and is to the Square of the whole Diameter, as 3 to 4.

Fig. 14.

Let there be drawn the Semidiameter AD. The Square of FD is equal to $FAq + DAq$ + the Rectangle FAX twice taken (*per* 12. *l.* 2.) But the Rectangle FAX twice taken is equal to the Square of the Semidiameter FA or DA: (For, becauſe AX, XB[d] are equal, the Rectangle FAX, twice taken, is equal to the two

[d] *Per Corol.* 5. foregoing.

two Rectangles which are under F A, A X, and under F A and X B, that is, equal to the Rectangle F A B [e]; that is, equal to F A q.) Therefore the F D q is treble to F A q or D A q the Square of the Semidiameter.

Now, becauſe the Square of the whole Diameter is quadruple of the Square of F A the Semidiameter [f], it is manifeſt that the Square of F D is to the Square of the Diameter, as 3 to 4.

Hence it follows that a Side of an equilateral Triangle is to the Diameter, as the ſquare Root of 3 is to 2, the ſquare Root of 4; and therefore that thoſe Lines are incommenſurable.

[e] *Per* 1. *l.* 2.

[f] *Per Corol.* 3. *Prop.* 4. *l.* 2.

PROP. XVI. Problem.

To inſcribe a regular Quindecagon in a Circle.

Fig. 15.

Inſcribe in the Circle A C the Side of a Pentagon [g], and A D the Side of an equilateral Triangle, (*per Corol.* 4. *p.* 15. *l.* 4.) biſect the Arch C D in E. C E is the Side of the Quindecagon, or fifteen-angled Figure ſought.

For, if the whole Circumference be ſuppos'd to be 15, the Arch A C will be 3, and the Arch A D 5, and therefore the Arch C D 2, and conſequently C E 1.

[g] *Per* 11. *l.* 4.

Corollary.

Fig. 15.

By this Method innumerable regular Figures may be inſcrib'd in a Circle. For if A C, A D, the Sides of two regular Figures, be inſcrib'd in a Circle, the Difference of the Arches C D will contain ſo many Sides of a new regular Figure, as are the Unites whereby the Denominators of the former differ one from another. But the Denominator of the new Figure is had, if the Denominators of the former be multiplied one by the other.

As if A D be the Side of a Square, and A C of a Decagon, the Difference of the Denominators is 6. Therefore the Arch C D contains 6 Sides of a new Figure. But the new Figure is of 40 Sides. For the Denominators 4 and 10 multiplied one by the other make 40.

Scholium.

THERE hath not yet been found out the Art by which regular Figures of 7, 9, 11, 13, 17, *&c.* Sides may be inſcribed in a Circle, by a Pair of Compaſſes and a Rule only. Foraſmuch as that Inſcription of Figures depends upon the Diviſion of the Circumference into any given Parts, which thing is yet lacking. But, if the Circumference of a Circle be divided into 360 Parts, you may in a mechanical Way inſcribe any regular Figures whatſoever in it, after this manner.

Problem 1.

Fig. 15. DIVIDE 360 Degrees (that is, the whole Circumference) by the Denominator of the Polygon to be inſcrib'd (*e. g.* a Nonangle.) Make at the Centre the Angle A G K of ſo many Degrees as are the Unites of the Quotient in the ſaid Diviſion. A K ſhall be the Side of the nine-angled Figure, which is required to be inſcrib'd in the Circle.

Problem

Problem 2.

BUT upon a given right Line you may deſcribe any regular Figure whatſoever by the Help of the following Table. Fig. 15.

A right Angle is to the Angle of the Figure,

		Difference.
In a Pentagon as	5 to 6 —	1
In an Hexagon as	3 to 4 —	1
In an Heptagon as	7 to 10 —	3
In an Octagon as	2 to 3 —	1
In a Nonagon as	9 to 14 —	5
In a Decagon as	5 to 8 —	3
In an Undecagon as	11 to 18 —	7
In a Duedecagon as	3 to 5 —	2

Let a regular Heptagon be to be inſcrib'd upon the given right Line XB. From the Centre X with the Interval XB deſcribe a Circle, from which cut off the Quadrant BO. See in the Table what is the Proportion of a right Angle to the Angle of an Heptagon: You will find it to be as 7 to 10, and the Difference is 3. Divide the Quadrant therefore into ſeven equal Arches, ſo many of which add to it from O to N as the Difference hath Unites. Thro' the three Points B, X, N, deſcribe (*per* 5. *l.* 4.) a Circle. This contains an Heptagon of the given right Line XB. Fig. 15.

The Table was made by means of Theorem 2. in the *Schol.* upon *p.* 32. *l.* 1. by which is found the Number of right Angles, which the Angles of any right-lin'd Figure make; which Number, being divided by the Denominator of the Figure, gives the Denominator of the Proportion of the Angle of the Figure to a right one.

Now, becauſe hitherto many things have been propounded concerning regular Figures, let the following famous Theorem of *Proclus* cloſe this Book.

Theorem.

ONLY three regular Figures, to wit, 6 equilateral Triangles, 4 Squares, and 3 Hexagons, can fill a Space; that is, can conſtitute one continued Superficies. Which is thus demonſtrated. That ſome regular Figure often repeated ſhould be able to fill a Space, it is required that the Angles of many Figures of that kind, being diſpoſed about one Point, ſhould make juſt four right ones; for juſt ſo many right Angles may be placed about one Point, as appears from *Corol* 3. *Prop.* 13. *l.* 1. As for Example, that equilateral Triangles ſhould fill a Space, it is requir'd that ſo many Angles of ſuch Triangles N, M, L, K, I, H, being diſpos'd about the Point A, ſhould make juſt four right ones. But ſix Angles of an equilateral Triangle do make four right ones; (for one makes two Thirds of one right one[h], and therefore ſix of them make 12 Thirds of one right one, that is, 4 right ones): Likewiſe the four Angles of a Square make four right ones, as is manifeſt; likewiſe three Angles of an Hexagon; for one maketh four Thirds of one right Angle (*per Corol.* 2. *p.* 15. *l.* 4.); and therefore three of them do make twelve Thirds of one right Angle, that is, four right ones. Therefore, *&c.*

Fig. 13.

[h] *Corol.* 12. *p.* 32. *l.* 1.

But that no other Figure beſides theſe can do this, will manifeſtly appear, if, its Angle being found as above, you ſhall multiply the ſame by any Number whatſoever; for the Angles will always either fall ſhort of, or exceed four right ones.

THE ELEMENTS OF EUCLID.

BOOK V.

THIS Fifth Book of Elements is altogether neceſſary for demonſtrating the Propoſitions of the Sixth Book. The Doctrine which it containeth is almoſt in continual Uſe. The Way of Reaſoning from Geometrical Proportion is moſt ſubtle, ſolid, and brief. This Method of Reaſoning, as a kind of Mathematical Logic, Geometry, Arithmetic, Muſic, Aſtronomy, Staticks, and all the other Parts of the Mathematicks, make eſpecial uſe of: Foraſmuch as they almoſt wholly depend upon Proportions connected together one with another; and are wont to borrow their Ways of Reaſoning concerning Proportionals from this Fifth Book. Practical Geometry, which conſiſts in the meaſuring of

Lines, Figures, and Solids, is for the moſt part derived from the Doctrine of Proportions. There is not a Rule in Arithmetic but what may be demonſtrated from the Propoſitions of this Fifth Book, without the Help of the 7th, 8th, and 9th Books, which treat profeſſedly of Numbers. We may fitly call the Muſic of the Ancients, Geometrical Proportions apply'd to tuneful Sounds; which ſame Thing you may well nigh ſay concerning Staticks, which are converſant about the Weights of Bodies. To comprehend the whole Matter in few Words, If you take away the Doctrine of Proportion from the Mathematicks, you will leave almoſt nothing which is excellent or greatly to be accounted of.

Scholium.

There is no Mathematician who is ignorant of how great Importance in Geometry the Knowledge of Proportions is; for it is the very Marrow, as it were, of the Mathematical Sciences; and the various Ways of Reaſoning concerning Proportionals are both moſt uſeful and moſt certain; neither can we without them move one Step.

But then I reckon that this Doctrine is congenite in Men's Minds with common Reaſon itſelf; and that the various Ways of Reaſoning concerning Proportionals, which Euclid *by much winding and going about delivers in this whole Book, do not ſo much need Demonſtration, properly ſo call'd, as Illuſtration and Examples. And I altogether am of Opinion, that thoſe, who take in Hand to deliver this moſt eaſy Doctrine by a long Circuit of Propoſitions, do involve a Thing in itſelf moſt clear, in a certain Cloud, and render it far more difficult. The Sum of the Matter I will open in a few Words. It is a Thing eaſily known, that* *four Quantities are then proportional, or that the Analogies are then alike, when the firſt Quantity contains the ſecond, as often as the third contains the fourth; or when the firſt is as often contain'd by the ſecond, as the third is by the fourth.* So 16 : 8 :: 4 : 2. *And* 3 : 9 :: 4 : 12. *Here are like or the ſame Proportions; becauſe in the former Example the Conſequents* 8 *and* 2 *are contain'd twice in their reſpective Antecedents; and ſo the Proportion of the Antecedents to the Conſequents is double. And in the other Example the Proportions are alſo alike; becauſe the Conſequents* 9 *and* 12 *do contain their reſpective Antecedents three Times; and ſo the Proportion of the Antecedents to the Conſequents is ſubtriple. (Nor is there any Proportion of commenſurable Quantities which may not be expreſſed by certain Numbers; nor indeed of Incommenſurables, which may not be expreſſed by Numbers infinitely approaching nearer and nearer unto the true one.) Furthermore, from what has been ſaid it appears, that like Proportions, whatſoever they are, may be expreſſed not only by divers Numbers, but alſo by the ſame. Thus* 2 *to* 1 *deſigns as well the Proportion of* 16 *to* 8, *as of* 4 *to* 2. 1 *to* 3 *no leſs expreſſeth that of* 4 *to* 12, *than that of* 3 *to* 9, *as is moſt manifeſt. Suppoſing therefore four Quantities to be proportional,* $A : B :: a : b$. *It is enquir'd in this Book after how many like Manners the Terms of theſe like Proportions may be changed, and ordered amongſt themſelves. So that the emerging Proportion on both Sides may be ſtill alike. And it may be anſwered, that it may be done after all the Ways and Manners poſſible; for ſeeing the Proportion of* A *to* B, *and that of* a *to* b *are alike, both of them may be expreſſed by the* ſame *Numbers after this Manner,* $A : B :: 9 : 3$, *and* $a : b :: 9 : 3$. *And*

And conſequently all the Proportions emerging on both Sides, either by Alternating *the Terms, or by* Inverting *them, or by* Compounding, *or* Dividing, *or* Converting, *or* Mixing *them, may be expreſſed by the* very ſame *Numbers*; *and conſequently the ſame Proportion will always be kept on both Sides. As, for Example ſake,* $A + B : B :: a + b : b$, *becauſe* $9 + 3 : 3$ *expreſſeth the ſame Proportion*; *which is* Compoſition. *The ſame is to be ſaid of all the Ways of changing the Terms. Therefore let Beginners obſerve this one Thing, that Proportions which are on both Sides the ſame be ever changed and ordered in the very ſame Manner. And then there will be no room to queſtion, whether the Proportions which ariſe on both Sides be alike or no. It is indeed a Thing to be wonder'd at, that no one of thoſe who have hitherto compiled Elements of Geometry, has made uſe of this moſt eaſy Method of ſtating the Equality of Proportions, for the illuſtrating of this Fifth Book about the Doctrine of Proportions. Take therefore the primary Ways which Geometry makes uſe of, in Reaſoning concerning like Proportions, as they are digeſted into this ſhort Table.*

Let it be	$A : B$	$:: a : b$	$:: 9 : 3$
Then it will be by			
Alternating	$A : a$	$:: B : b$	~~:: 9 :~~ 9 :: 3 : 9
Inverting	$B : A$	$:: b : a$	$:: 3 : 9$
Compounding	$A + B : B$	$:: a + b : b$	$:: 9 + 3$ (12) $: 3$
Dividing	$A - B : B$	$:: a - b : b$	$:: 9 - 3$ (6) $: 3$
Converting	$A : A + B$	$:: a : a + b$	$:: 9 : 9 + 3$ (12)
or	$A : A - B$	$:: a : a - b$	$:: 9 : 9 - 3$ (6)
Mixing	$A + B : A - B$	$:: a + b : a - b$	$:: 9 + 3 : 9 - 3$
Ex æquo	$A : B :: a : b$, & $B : C :: b : c$, *then* $A : C :: a : c$.		
	$9 : 3 :: 9 : 3$, $3 : 1 :: 3 : 1$, *then* $9 : 1 :: 9 : 1$.		
Ex æquo	$A : B :: a : b$, & $B : C :: r : a$, *then* $A : C :: r : b$.		
perturbatè	$8 : 3 :: 8 : 3$, & $3 : 12 :: 2 : 8$, *then* $8 : 12 :: 2 : 3$.		
Or thus,	$8 : 3 :: 16 : 6$, & $3 : 2 :: 24 : 16$, *then* $8 : 2 :: 24 : 6$.		
	$a : b :: e + a : a + b$ & $b : e :: a + b : a + e$ *then* $a : e :: a + b : e + b$		

He therefore that is expert in theſe Ways of Reaſoning concerning Proportionals, and knows how to bring them into Uſe upon Occaſion, will ſeldom ſtand in need of the particular Propoſitions of the Fifth Book. Only two of them, which yet are almoſt Axioms, may not improperly be inſerted and illuſtrated by Examples, in way of an Appendix, *becauſe of the Frequency of their Uſe in all the Parts of the Mathematicks; which therefore ſhall be done after the Definitions.*

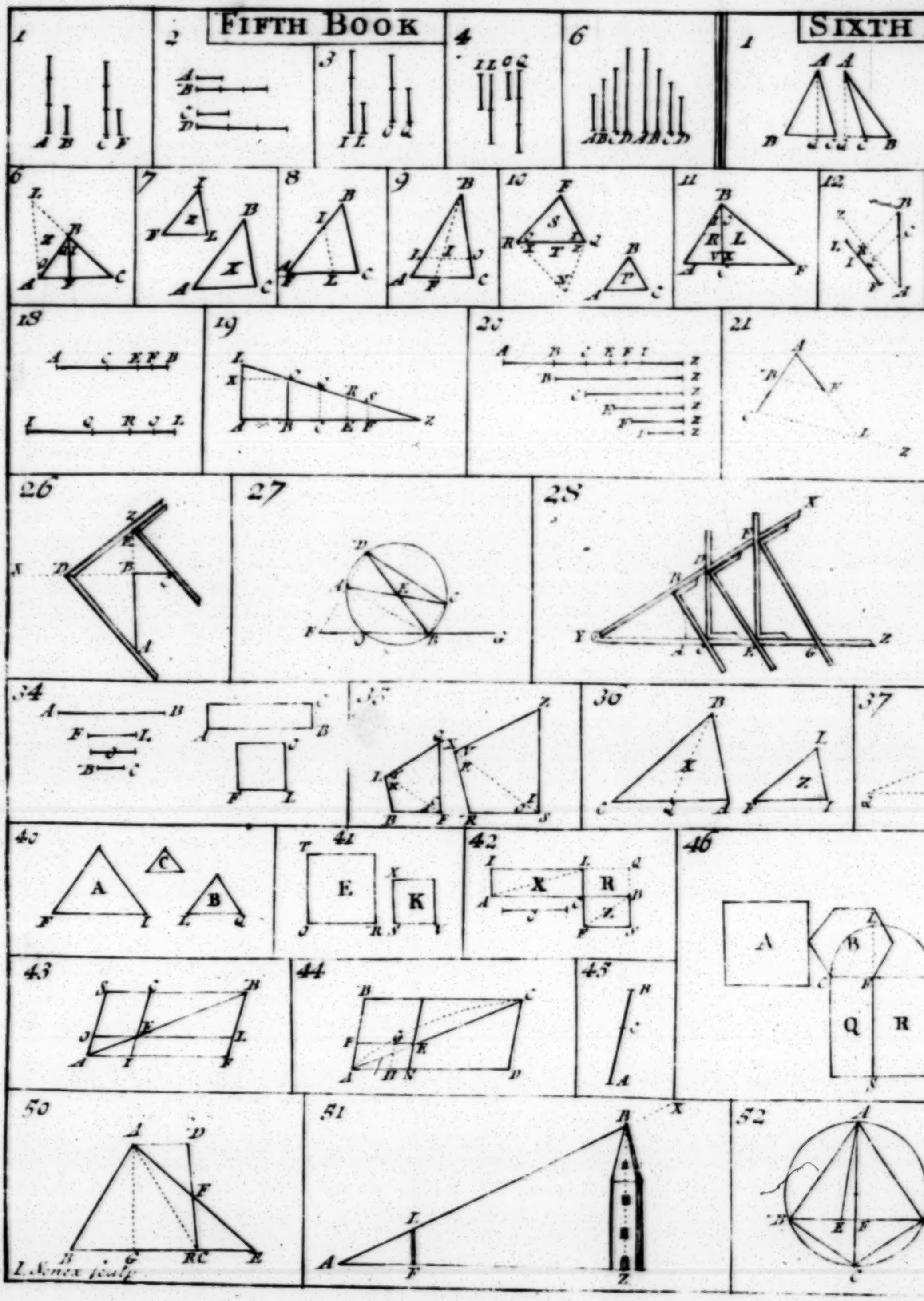
DEFINITIONS.

1. AN *Aliquot* Part of Magnitude is that which, being ſo many Times more or leſs repeated, doth meaſure or is juſt equal to the Magnitude. An *Aliquant* Part is that which doth not meaſure it.

The Length of one Foot is an *Aliquot* Part of the Length of 10 Feet, becauſe being ten Times repeated it meaſures it. But the Length of four Feet is an *Aliquant* Part of a Line of 10 Feet, becauſe being ſo many Times repeated, to wit, twice, it falls ſhort of it, but being thrice repeated it exceeds it.

2. One Magnitude is ſaid to be a *multiple* of another, when the leſſer meaſures the greater, and conſequently is an Aliquot Part thereof; or when the greater contains the leſſer ſo many Times preciſely.

3. Proportion is the mutual Reſpect, as to Quantity, of two Magnitudes of the ſame Kind.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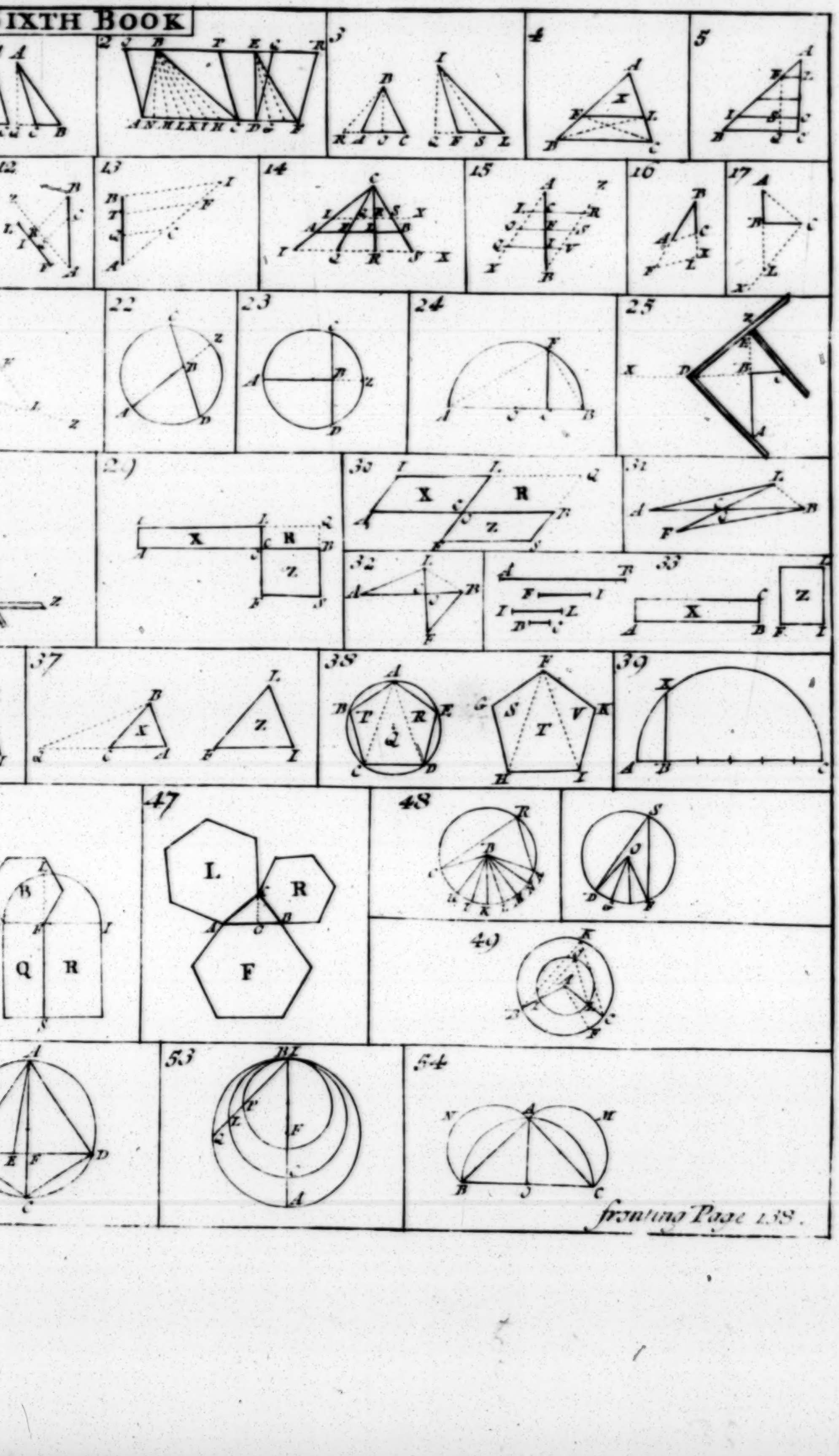

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Therefore there are in all Proportions two Terms, of which that is called the Antecedent which is firſt named, or which is nam'd in the Nominative Caſe; the other the Conſequent.

When the Antecedent and the Conſequent are equal, it is called Proportion of Equality; when they are unequal, Proportion of Inequality.

4. Rational Proportion is that which is betwixt commenſurable Magnitudes, and may be expreſſed in Numbers. Irrational Proportion, that which is betwixt Quantities incommenſurable, and cannot be explicated by any Numbers.

Moreover, Commenſurable Quantities are thoſe which ſome common Meaſure meaſureth; Incommenſurable, thoſe which cannot be meaſured by any common Meaſure.

5. Two Proportions (that of A to B, and that of C to D) are alike, equal, or the ſame; when the Antecedent of one (A) doth equally or in the ſame Manner (that is, neither more nor leſs, contain its Conſequent (B), as the Antecedent of the other (C) contains its Conſequent (*F*). Fig. 1. *l.* 5.

Or when the Antecedent of the one (A) is ſo often contain'd in its Conſequent (B), as (C) the Antecedent of the other is in its Conſequent (D). Fig. 2.

6. Two Proportions are unlike, or one is greater than the other, when the Antecedent of one (I) doth more contain its Conſequent (L), than the Antecedent of the other (O) doth contain its Conſequent (Q); or when the Antecedent of one is leſs contain'd in its Conſequent, than the Antecedent of the other in its Conſequent. Fig. 3. Fig. 4.

7. Like or *ſimilar* Parts are thoſe which are equally or in the ſame Manner contain'd in their Wholes; ſo that what Sort of Part one is of its Whole, ſuch a Part the other is of its Whole. Which Thing indeed is nothing elſe, but that the Parts bear the ſame Proportion to their Wholes.

Aliquot Parts are like, which do equally meaſure their Wholes, as if each of them be one Third or one Tenth, *&c.* of its Whole.

Fig. 6. 8. Magnitudes (A, B, C, D) are ſaid to be *continually* proportional when the middle Terms (B, C) are taken twice; that is, when they are each of them a Conſequent in reſpect of the foregoing, and an Antecedent in reſpect of the following.

We thus pronounce continual Proportions, A is to B as B to C; and B is to C, as C is to D. And ſo on.

9. Magnitudes are *diſcretely* proportional when no Term is twice taken.

Fig. 1. Diſcrete Propoſitions we thus pronounce, A is to B as C to F. When there are more than three proportional Magnitudes, if they be ſaid to be proportional, they are always underſtood to be diſcretely ſo.

Fig. 6. 10. When the Magnitudes (A, B, C, D) are continually proportional, the firſt (A) is ſaid to have to the third (C) a duplicate Proportion of that which it hath to the ſecond (B): And the firſt (A) is ſaid to have to the fourth (D) a triplicate Proportion of that, which the ſame firſt hath to the ſecond (B): And ſo forwards.

[*If one triplicate Proportion be equal to another duplicate Proportion, the latter ſimple Proportion ſhall be ſeſquiplicate, or one and a half of the former ſimple Proportion. Let A, B, C, D, be* *be ∺ ; and a, b, c, ∺ ; and let A the firſt in the former Analogy be unto D the fourth, as (a) the firſt in the ſecond Analogy is to (c) the fourth ; I ſay that (a) is to (b) in a Proportion which is one and a half of that which A is in to B. For let F be a middle Proportional betwixt B and C : Or, which is the ſame thing, betwixt A and D. Becauſe of the Equality of the Proportions of A to D, and (a) to (c), and the middle Proportions on both Sides F and (b) ; it will be A : F : : a : b. But the Proportion of A to F is compounded of the entire Proportion of A to B, and of the Proportion of the ſame B to C halved ; and conſequently the Proportion of (a) to (b), which is equal to that of A to F, contains the entire Proportion of A to B, and alſo the ſame halv'd, to wit, the Proportion of B to F. But the whole Proportion, with its half, is a ſeſquiplicate or ſeſquialteral Proportion, or that which is one and a half of the other. (a) therefore is to (b) in a Proportion ſeſquiplicate of that of A to B. So in Aſtronomy, ſince the Cubes of the Diſtances of the Planets from the Sun bear that Proportion one to another, which the Squares of their periodical Times bear ; ſo that the triplicated Proportion of the Diſtances is the ſame with the Duplicate one of the periodical Times ; it is wont to be ſaid, that the periodical Times are in a ſeſquiplicate or ſeſquialteral Proportion of their Diſtances from the Sun.*]

11. Antecedent Magnitudes are ſaid to be Homologous or like to Antecedent, and Conſequent to Conſequent Magnitudes. As if A Fig. 1. is to B, as C to F ; A, C, and B, F, are homologous Quantities.

XII. If a Set of Pairs of Quantities contain every one the ſame Proportion, that is the very

Proportion alſo which the Sum of all the Antecedents bears to the Sum of all the Conſequents.

$$\frac{20}{10}+\frac{6}{3}+\frac{8}{4}+\frac{18\cancel{4}}{9}+\frac{14}{7}=\frac{66}{33}$$

XIX. If Parts be as Wholes, the Remainders will be alſo in the very ſame Proportion.

If 30 be to 20, as 3 to 2; 27 will be to 18 alſo, as 30 to 20, or as 3 to 2.

THE

THE ELEMENTS OF EUCLID.

BOOK VI.

THE Doctrine of Proportions, which was generally ſet forth in the Fifth Book, is applied in the Sixth, to plain Figures. And thoſe Things which are delivered in this Book are ſo neceſſary to be known, that without them no Man can penetrate into the Secrets of Geometry, and reap the ſweet Fruits of the Mathematicks. Each Propoſition deſerves to have an Encomium annexed; ſo great is the Utility of all.

This Sixth Book, as hath been ſaid, begins to apply that excellent Doctrine concerning Geometrical Proportion, which was juſt before delivered, to divers

divers and, thoſe certainly, moſt notable Uſes; and, beginning with Triangles, the moſt ſimple of Figures, ſearches out their Sides and Areas, as they anſwer to one another in a certain Proportion. Then it defines proportional Lines, and the proportional Augmentations or Diminutions of Figures; and ſhews in what manner we may either increaſe or diminiſh them according to any Proportion given. It opens likewiſe the Golden Rule, or Rule of Proportion, the very chief of all Arithmetic; and demonſtrates that in a rectangle Triangle, not only the Square, but alſo the Pentagon, Hexagon, and, in general, every regular Polygon, which is deſcribed by the Hypotenuſe, is equal to the Squares, Pentagons, Hexagons, or any regular Polygons whatſoever, that are deſcrib'd by the two Sides. It alſo propounds moſt eaſy and certain Principles for meaſuring as well Solids, as Lines and Surfaces, which are of very great Uſe in all Parts of the Mathematicks.

DEFINITIONS.

I. LIKE or ſimilar Figures are thoſe which both have all the Angles equal, each to each other reſpectively, and the Sides which are oppoſed to the equal Angles or which are betwixt them, or which are about the equal Angles (for they come all to one) proportional.

Fig. 7. *l.* 6. As the Triangle X, Z, will be ſaid to be like, or ſimilar, if the Angle A be equal to the Angle F, and the Angle B equal to the Angle I, and conſequently the Angle C equal to the Angle L: and moreover if AB be to FI, as BC to LI; and BC is to LI, as CA is to LF; and CA to LF as AB to FI; by comparing always the Sides oppoſite to the equal Angles. In the ſame manner the Likeneſs of all right-lin'd Figures may be explain'd.

2. Reciprocal Figures are when the Antecedent and Conſequent Terms of the Proportions appear on both Sides. Fig. 29.

As in the Parallelograms X, Z,
If AC be to CB,
As FC is to CL.

The Antecedents here are AC, and FC, of which there is one in both Figures; and the Conſequents are CB, and CL, of which likewiſe there is one in each Figure. And therefore the Parallelograms X, Z, are call'd reciprocal. Underſtand the ſame of other Figures.

3. The Altitude of a Figure is the Perpendicular let fall from the Top to the Baſe. This is with *Euclid* the fourth Definition.

As the Altitude of the Triangle ABC is the Perpendicular AQ which falls from the Top upon the Baſe BC, either within the Triangle or without, upon the Baſe protracted. Now the Baſe and Top are aſſumed at Pleaſure. Fig. 1. 3

4. Like Arches of Circles are thoſe which have the ſame Proportion unto their whole Circumferences

As if each of them be a third or fourth Part, &c. of that Circumference.

PROPOSITION I. Theorem.

TRIANGLES (ABC, DEF) and Parallelograms (AOPC, DQRF) Fig. 2.

which are betwixt the ſame Parallels, or have the ſame Altitude, have the ſame Proportion betwixt themſelves as their Baſes, (AC, DF.)

Upon this Theorem the whole Sixth Book depends, yea, whatſoever any where is demonſtrated about Figures by Proportions, whether Plain or Solid.

Let there be taken any Aliquot Part of the Baſe DF: *EG.* DG one Third, and let the right Line GE be drawn: The Triangle DEG will likewiſe be one third Part of the Triangle DEF, as is gathered from 38. l. 1. Wherefore DG and the Triangle DGE are like Aliquot Parts of their Conſequents[a]. Then let there be taken away DG from the Baſe AC as often as it can, as ſuppoſe ſix times, and let the right Lines HB, IB, KB, LB, MB, NB, be drawn. Becauſe the Lines CH, HI, *&c.* are each of them equal to DG, the ſix Triangles CBH, HBI, *&c.* are each of them [b] equal to the Triangle DEG. Therefore as often as DG is contain'd in the Antecedent AC, ſo often is the Triangle DEG contain'd in the Triangle ABC. By the ſame Reaſoning it may be ſhew'd, that the like Aliquot Parts whatſoever of the Conſequents (the Baſe DF, and the Triangle DEF) are in an equal Number contain'd in the Antecedents (the Baſe AC, and the Triangle ABC) therefore as the Baſe AC is to the Baſe DF; ſo is the Triangle ABC to the Triangle DEF. *Q. E. D.*

But now becauſe the Parallelograms AP, DR, are [c] double to the Triangles ABC, DEF, they alſo will be as their Baſes.

Corollary.

[a] *Per Def.* 7. *l.* 5.

[b] *Per* 38. *l.* 1.

[c] *Per* 41. *l.* 1.

Corollary.

THE Triangles (ABC, FIL) and the Parallelograms which have equal Baſes (AC, FL) or the ſame, have that Proportion one to another, which their Altitudes (BO, IQ) have. Fig. 3.

For let QS, OR, be made equal to the equal Baſes (FL, AC); QS, OR will then be equal. Draw SI, RB. If in the Triangles OBR, QIS, you take BO, IQ for the Baſes, OR, QS, will be their Altitudes; which ſeeing they are equal, the Triangles OBR, QIS[d], will be betwixt themſelves, as their Baſes BO, IQ. But becauſe by the Conſtruction OR is equal to AC, and QS equal to FL, the Triangles OBR, QIS, are[e] equal to the Triangles ABC, FIL. Therefore the Triangles ABC, FIL, are alſo as BO is to QI.

[d] *Per* 1. *l.* 6.

[e] *Per* 38. *l.* 1.

Coroll. (2.) *Hence a Trapezium ABCD, whoſe Sides AD and BC are parallel, may be divided into any equal Parts whatſoever. For let CE be made equal to AD. Becauſe of the Equality of the Angles vertically oppoſite*[f] *AFD, EFC, and of the alternate Angles*[g] *DAF, FEC, and ADF, ECF, and the Equality of the Baſes AD, CE, by Conſtruction, the Triangles ADF, FCE*[h]*, are equal; and therefore the Triangle ABE is equal to the Trapezium ABCD. Therefore the Baſe BE being divided into any equal Parts whatſoever; as, for Inſtance, three, BG, GR, RE, the Triangles ABG, AGR, ARE, ſhall each of them be one third Part of the Trapezium.* Q. E. I. Fig. 50.

[f] *Per* 15. *l.* 1.

[g] *Per* 27. *l.* 1.

[h] *Per.* 26. *l.* 1.

PROP. II. Theorem.

Fig. 4.

IF to one Side of a Triangle (as BC) there be drawn (FL) a Parallel, this cuts the Sides proportionally, that is (AF) will be to (FB) as (AL) to (LC).

And if the right Line (FL) cuts the Sides (BA, CA) proportionally; it will be parallel to the other Side (BC).

Part 1. Let BL, CF be drawn. Becauſe FL is ſuppoſed parallel to BC, the Triangles FBL, LCF having the ſame Baſe are [i] equal. Therefore the Triangle X hath the ſame Proportion to both; now the Triangle X is to the Triangle FBL, as the ſame Triangle X is to that LCF. But the Triangle X is to the Triangle FBL [k], as AF is to FB; and the Triangle X is to that LCF as AL [l] is to LC. Therefore alſo AF is to FB, as AL to LC. *Q. E. D.*

[i] *Per* 37. *l.* 1.

[k] By the foregoing.

[l] By the ſame.

Part 2. As AF is to FB, ſo is the [m] Triangle X to the Triangle FBL: And as AL is to LC, ſo is the ſame Triangle X to the Triangle LCF. Now AF is ſuppos'd to be to FB, as AL is to LC. Therefore the Triangle X is to the Triangle FBL, as the ſame X is to LCF. Therefore the Triangles FBL, LCF are equal. Therefore ſeeing they have a common Baſe FL, the Lines FL, BC, are [n] parallel. *Q. E. D.*

[m] By the foregoing.

[n] *Per* 39. *l.* 1.

Corollary.

Fig. 5.

IF unto (BC) one Side of a Triangle there be drawn more Parallels (IO, FL) all the Segments of the Sides will be proportional.

Let FQ be drawn parallel to AC. The right Lines FS, SQ, are equal [o] to LO, OC. But BI is to FI as QS to SF [p]. Therefore BI is alſo to IF as CO to OL.

[o] *Per* 34. *l.* 1.

[p] *Per* 2. *l.* 6.

PROP. III. Theorem.

Fig. 6.

IF a right Line (BF) which biſects an Angle of a Triangle, doth alſo cut the Baſe (AC), the Segments of the Baſe (AF, FC) will have the ſame Proportion betwixt themſelves as the Sides (AB, BC) have.

And if the Parts of the Baſe (AF, FC) have the ſame Proportion betwixt themſelves, as the other Sides (AB, CB) the Line (BF) which cuts the Baſe, biſects the oppoſite Angle (ABC).

Part 1. Draw forth CB until BL be equal to BA, and join AL. Becauſe in the Triangle Z the Sides LB, AB, are equal, the Angles alſo [q] L and O are equal. Becauſe therefore the external Angle ABC is equal to the two internal ones [r] L, O, the Angle I, which by the Hypotheſis is half ABC, will be equal to the Angle L. Therefore AL, FB [s] are parallel. Therefore in the Triangle ACL, AF is to FC [t] as LB (that is, AB) is to BC. *Q. E. D.*

[q] *Per* 5. *l.* 1.

[r] *Per* 32. *l.* 1.

[s] *Per* 29. *l.* 1.

[t] *Per* 2. *l.* 6.

Part 2. Protract CB again until BL be equal to BA. Becauſe AF is ſuppos'd to be to FC, as AB (that is, LB) is to BC; AL, FB[u] are parallel. Therefore the external Angle I is equal to the internal one L[w]; and the alternate Q equal to the alternate O. But becauſe LB, AB, are equal, the Angles L and O[x] are equal. Therefore I and Q are alſo equal. Therefore ABC is biſected. *Q. E. D.*

[u] *Per* 2. *l.* 6.

[w] *Per* 27. *l.* 1.

[x] *Per* 5. *l.* 1.

PROP. IV. Theorem.

TRIANGLES *which are equiangular to one another are like or ſimilar, that is, have their Sides alſo*[y] *that are oppoſite to the equal Angles proportional.*

[y] *Per Def.* 1. *l.* 6.

Fig. 7.

In the Triangles X, Z, let the Angle A be equal to the Angle F, and the Angle C to the Angle L, and the Angle B to the Angle I; I ſay, that AB is to FI as AC is to FL; and AC is to FL as CB is to LI; and CB is to LI as BA is to FI.

Fig. 7, 8.

Demonſt. If the Angle F be laid upon its equal A, the Sides FI, FL, will fall upon the Sides AB, AC. And, becauſe the external Angle AIL is by the Hypotheſis equal to the internal B[z], therefore[a] IL, BC, are parallel. Therefore BI is to IA[b] as CL to LA. Therefore by compounding BA is to IF as CA to LF. And, if the Angle L be laid upon the Angle C, it will be ſhew'd in the ſame manner, that AC is to FL, as BC is to IL; and, if the Angle I be laid upon the Angle B, it will be ſhew'd in the ſame

[z] Fig. 8. alone.

[a] *Per* 29. *l.* 1.

[b] *Per* 2. *l.* 6.

manner, that BC is to IL as AB to FI. The Propoſition therefore is prov'd.

Corollaries.

Fig. 8.

1. IF in a Triangle a Line LI be drawn parallel to one Side BC, the Triangle LFI will be like to the Whole CBF; and conſequently CF will be to LF, as BC to LI.

For becauſe LI, BC, are parallel, the external Angles FIL, FLI, will (*per* 27. *l.* 1.) be equal to the internal ones B and C: But F is common to both Triangles. Therefore they are equiangular. Therefore the Sides CF, LF oppoſite to the equal Angles B and FIL [c] are proportional to the Sides BC, LI, which are oppos'd to the common Angle F.

Fig. 9.

2. If in a Triangle a right Line BF, drawn from the oppoſite Angle B, doth cut the Parallels AC, LO, it cuts them proportionally.

For by *Coroll.* 1. AF is to LI, as FB is to IB; and FC alſo is to IO, as FB is to IB. Therefore AF is to LI, as FC to IO. Therefore by changing AF is to FC, as LI to IO.

Fig. 51.

[3. *From Coroll.* 1. *we learn to find the Height of a Tower, or any elevated Point, by only the Shadow of a Staff. Fix the Staff FL perpendicularly upon the Ground in that Place where the Ray of the Sun XBA, that terminates the Shadow of the Tower BZ, may alſo paſs through L. There will be in the Triangle AZB the Line FL parallel to the Height of the Tower ZB. Whence as AF, the Diſtance of the Staff from the Point of the Shadow, is to FL the Length of the Staff; ſo is AZ, the Diſtance of the Tower*

[c] By the foregoing.

from the Point of the Shadow, to ZB *the Height of the Tower. And, becauſe the three firſt Terms are eaſily had by meaſuring, the fourth alſo,* i. e. *the Height of the Tower, is had alſo.* Q. E. I.

Fig. 52. 4. *From this alſo incomparably uſeful Propoſition, we may deduce that famous Theorem of* Ptolemy; *to wit, that in every Quadrilateral inſcrib'd in a Circle, the Rectangle of the Diagonals* $AC \times BD$ *is equal to the two Rectangles of the oppoſite Sides,* $AB \times CD$ *and* $AD \times BC$. *For let the Angle* BAE *be made equal to the Angle* CAD. *Becauſe the Angles* BAE, CAD, *are equal by Conſtruction, the Angles* ABE, ACD, *ſtanding upon the ſame Arch* AD, *are*[d] *equal; therefore the Triangles* BAE, CAD, *are alike. And* $AC : CD :: AB : BE$; *and conſequently*[e] *the Rectangle of the Extremes* $AC \times BE$ *is equal to the Rectangle of the Means* $CD \times AB$. *In like manner, becauſe the Angle* EAD *is equal to* BAC *by Conſtruction, and the Angles* ADE, ACB, *as ſtanding upon the ſame Arch* AB, *are equal: The Triangles* ADE, ACB, *will be like; and* $AD : DE :: AC : CB$. *And therefore the Rectangle of the Extremes* $AD \times CB$ *is equal to the Rectangle of the Means* $DE \times AC$. *But the Rectangles* $AC \times BE$, *and* $AC \times DE$, *are equal to the Rectangle* $AC \times BD$. *Therefore the Rectangles* $AB \times DC$, *and* $AD \times BC$, *which are made by the oppoſite Sides, are equal to the Rectangle* $AC \times BD$, *which is made by the Diagonals.* Q. E. D.]

[d] Per 21. *l.* 3.

[e] Per 16. *l.* 6.

PROP. V. Theorem.

Fig. 10. I*F two Triangles have all their Sides proportional each to each, they ſhall alſo be mutually equiangular.*

That is, If AB be to RF, as AC to RQ; and as AC is to RQ, ſo is CB to QF; and as CB is to QF, ſo is AB to RF; I ſay, that the Angles oppoſite to the Antecedents are equal to the Angles oppoſite to the Conſequents; to wit, C to I, and B to F, and A to O.

Ang.	Antec.	Conſeq.	Ang.
C	AB	RF	I
B	AC	RQ	F
A	CB	QF	O

Make X and Z equal to A and C; and let the Sides meet in N. The Angles B and N will [f] be alſo equal. Becauſe therefore the Triangles P, T, are equiangular, AB (by the foregoing) will be to RN, as AC to RQ. But, by the Hypotheſis, AB is to RF as AC to RQ. Therefore AB is to RF as the ſame AB is to RN. Therefore RN, RF, are equal. In the like manner I might ſhew that QN and QF are equal. Therefore the Triangles T, S, are equilateral to each other. Therefore the Angles I, F, O, are equal (*per* 8. *l.* 1.) to the Angles Z, N, X, that is, by the Conſtruction to the Angles C, B, A. *Q. E. D.*

[f] *Per Corol.* 9. *p.* 32. *l.* 1.

PROP. VI. Problem.

IF two Triangles (P, S) have one Angle (A) equal to one Angle (O); and the Sides (AB, AC, RF, RQ) which contain the equal Angles proportional; the Triangles will be like.

Let X and Z be made equal to the Angles A, C, and the Sides meet together in N. Therefore the Angles B and N will[g] be alſo equal. Then it may be ſhew'd, as in the foregoing, that R F, R N, are equal. But R Q is common to both Triangles S, T. The Angles alſo O and X are equal, becauſe they are both equal to the ſame A, the one X by the Conſtruction, and O by the Hypotheſis. Therefore[h] I and F are likewiſe equal to Z and N. Therefore the Triangle S is equiangular to the Triangle T; that is, by the Conſtruction to the Triangle P. Therefore S, P are alike, (*per* 4. *l.* 6.) *Q. E. D.*

[g] *Per Corol.* 9. *p.* 32. *l.* 1.

[h] *Per* 4. *l.* 1.

PROP. VII.

IS *ſcarce of any Uſe.*

PROP. VIII. Theorem.

Fig. 11.

IN *a Rectangle Triangle, the Perpendicular (B C) let down from the right Angle to the Baſe, cuts the Triangles into Parts like to the Whole, and betwixt themſelves.*

In the Triangles A B F and L, the Angle F is common, but the Angles A B F and X are by the Hypotheſis right ones, and conſequently equal. Therefore the other Angles A and O are[i] alſo equal. Therefore[k] the Triangles A B F and L are alike. In the ſame Manner the Triangles A B F and R may be ſhew'd to be equal, and the Angle I equal to the

[i] *Per Corol.* 9. *p.* 42. *l.* 1.

[k] *Per* 4. *l.* 6.

the Angle F. From which it is now manifeſt, that R and L alſo are alike, ſeeing the Angles I and F; O and A; U and X are equal. *Q. E. D.*

Corollaries.

FIRST, BC is a mean Proportional betwixt AC, CF.

For ſeeing there be in the Triangle R and L

equ. Ang. I. F	equ. Ang. A. O
Sides oppoſ. AC. CB	Sid. oppoſ. CB. CF.

It is manifeſt[l] that AC : CB :: CB : CF.

2. BF is a mean Proportional betwixt AF and CF. Likewiſe AB a mean betwixt FA and CA.

For in the Triangle ABF and L.

equ. Ang. ABF. X	equ. Ang. A. O
Sid. oppoſ. AF. BF	Sid. oppoſ. BF. CF

Therefore AF[m] : BF :: BF : CF. Likewiſe becauſe in the Triangle ABF and R there be

equal Ang. ABF. V	equ. Ang. F. I
Sid. oppoſ. AF. AB	Sid. oppoſ. AB. AC

It will be again AF : AB :: AB : AC.

3. *Hence we learn to meaſure an inacceſſible Line, one Term whereof is acceſſible. Let the inacceſſible Line be CF. Let there be rais'd from the Point C the Perpendicular CB: And to any Point of this Perpendicular, as B, let there be applied a Square or any right Angle ABF; ſo that in looking along the Line BF the Point F, and along the Side BA the Point A may be obſerved. Let the acceſſible Line AC be meaſured, and from the following Analogy the inacceſſible CF will be made known. AC : CB :: CB : CF. Let the* Fig. 11.

Square

[l] *Per* 4. *l.* 6.

[m] By the ſame.

Square then of the Line CB be divided by the Line AC, and the Quotient[n] *will give the ſought Line CF.* Q. E. I.

[n] *Per Corol.* 3. *p.* 17. *l.* 6.

PROP. IX. Problem.

Fig. 12.

TO divide a given Line (AB) according to a given Proportion (FI to IL).

Let the infinite Line AZ be drawn. From which take AQ, QR, equal to FI, IL. From R draw RB. Parallel to this draw QC from Q. I ſay the Thing is done.

It is manifeſt from *Prop.* 2. *l.* 6.

PROP. X. Problem.

Fig. 13.

TO divide a given Line as (AB) in like manner as another given one (AI) hath been divided (in F, C).

Let the right Line IB join the Extremities of the two Lines. Draw Parallels to this from the Points F, C, which may meet the right Line, that is to be cut, AB in L and Q. I ſay the Thing is done.

This is manifeſt from the Corollary of *p.* 2. *l.* 6.

Fig. 53.

[*Or thus, if the cut Line IA be greater than that which is to be cut BQ, let three Circles touching one another be deſcrib'd with the Diameters IF, IC, IA; and let the Subtenſe BQ be fitted from the Point I to the Circumference of the greateſt Circle: The two leſſer Circles will cut the Line BQ in the Points L, P, in the Proportion*[c] *of the Sections of the Diameter LA. If the Line IA be*

[c] *Per Corol.* 4. *p.* 31. *l.* 3.

be cut into four Parts, four Circles are to be drawn; if into five, then five Circles; and ſo infinitely.]

Scholium.

FROM this Propoſition we learn to cut a right Line given into any equal Parts whatſoever. Let an infinite right Line make any Angle with the right Line to be cut A B; from which take with a Pair of Compaſſes ſo many equal Parts A C, C F, F I, as you would divide A B into. Draw the right Line I B, and the Parallels to it F L, C Q. I ſay the Thing is done. Fig. 13.

We may do the ſame Thing otherwiſe, and more eaſily after *Maurolycus*, in the Manner following. Let A B be to be triſected or divided into three equal Parts. Draw the infinite Line I X parallel to A B, above or below it. From I X, if it be below A B, take with a Pair of Compaſſes three equal Parts I Q, Q R, R S, which together may be greater than A B; but leſſer if I X is above. Through I and A, as likewiſe through S and B, draw right Lines which may meet together in C. From C to Q and R draw right Lines: Theſe will triſect the given Line A B. The Demonſtration appears from *Coroll.* 2. *Prop.* 4. Fig. 14.

Again, with *Maurolycus* we may otherwiſe obtain the ſame Thing, to wit, thus: Let A B be to be quadriſected. Draw the infinite Line A X and B Z alſo an infinite Line parallel to it. From theſe take with the Compaſſes equal Parts A L, L O, O Q, and B V, V S, S R, in each fewer Parts by one, than are required in A B; then let there be drawn the right Lines, L R, O S, Fig. 15.

O S, Q V. Theſe will quadriſect the given A B.

For becauſe, by the Conſtruction, the Lines L O, R S, parallel and equal, are join'd by L R and O S, theſe alſo [p] will be parallel. In the like manner O S and Q V are parallel. Therefore ſeeing A Q is cut into three equal Parts, A I will alſo [q] be cut into ſo many equal Parts. Likewiſe B C will be cut into three equal Parts. Therefore the Whole A B will be cut into four equal Parts.

[p] *Per* 33. *l.* 1.

[q] *Per Corol. Prop.* 2. *l.* 6.

Theſe two Ways of Practice are eaſier than *Euclid*'s, becauſe fewer Parallels are to be drawn.

PROP. XI. Problem.

Fig. 16.

TO find a third Proportional to two right Lines given (A B, B C).

Draw the right Line A C. From B A produced take A F equal to B C. Through F draw the infinite Line F X parallel to A C, which Infinite let B C produc'd meet in L. I ſay that A B is to B C, as B C to C L.

For A B : A F [r] :: B C : C L. But A F [s] is equal to B C. Therefore A B : B C :: B C : C L; and ſo C L is the third Proportional ſought.

[r] *Per* 2. *l.* 6.

[s] By the Conſtruction.

Otherwiſe.

Fig. 17.

LET A B and B C be ſet at a right Angle. Join A C. From C draw C X perpendicular to A C infinite; which C X let A B produc'd meet in L. I ſay A B : B C :: B C : B L. It is manifeſt from *Coroll.* 1. *pr.* 8.

Scholium.

Scholium.

A Given Proportion may not only be continued in three, but alſo in infinite Terms, and the whole Sum of the infinite proportional Terms be exhibited. *Gregory* of St. *Vincent* hath very handſomely proſecuted this Matter, and the whole Buſineſs of Geometrical Progreſſion, in the whole ſecond Book of his Work. We, for the Sake of the Studious, will here preſent ſuccinctly the Conſtruction and Demonſtration of the Thing propoſed.

Problem.

LET a Proportion of the greater Inequality be given, as A B to B C. It is requir'd to continue this through infinite Terms, and to preſent the Sum of them all. Fig. 19.

Let the Perpendiculars A L, B O, be erected, and taken equal to the given Lines A B, B C, and through L, O, let a right Line be drawn, meeting with A B C produc'd in Z. I ſay, 1. If from C you erect the Perpendicular C Q; C Q ſhall be a third Proportional. Transfer Q C into C E, and from E erect E R; this ſhall be a fourth Proportional. Transfer E R into E F, and erect F S; this ſhall be a fifth Proportional; and ſo the Proportion of A B to B C, that is, of A L to B O, will be continued through the Terms, A L, B O, C Q, E R, F S, *&c.* or A B, B C, C E, E F, F I, *&c.* infinitely, becauſe every Term (as F S) may be taken away from the remaining one

[t] *Per Corol.* 1. *Prop.* 4. *l.* 6.

one F Z; for ſeeing L A (that is, A B) is leſs than A Z; F S alſo [t] muſt ever be leſs than F Z.

I ſay, 2. A Z is equal to the whole Sum of the infinite Proportionals.

Part I. [*It being ſuppos'd as before,* $AZ : BZ :: AB : BC$; *it will be by alternating* $AZ : AB :: BZ : BC$. *And by dividing* $AZ - AB : AB :: BZ - BC$; *that is,* $BZ : AB :: CZ : BC$. *Therefore by inverting* $AB : BZ :: BC : CZ$. *And by compounding* $AB + BZ : BZ : BC + CZ : CZ$; *that is,* $AZ : BZ :: BZ : CZ$.] But as A Z is to B Z, ſo is L A to O B; and as B Z is to C Z ſo is O B to Q C. Therefore alſo L A is to OB, as OB is to Q C. In the ſame Manner I might ſhew that OB is to Q C, as Q C to R E; and ſo forwards infinitely.

Part II. The whole Sum of the infinite Terms is neither leſs than A Z, nor greater; therefore it is equal. It is not greater, becauſe ſeeing we have ſhew'd above, that Q C is leſſer than C Z, and R E than E Z, and S F than F Z, and ſo on infinitely, all the Terms Q C, R E, S F, *&c.* may be infinitely ſet one by another in the right Line A Z, ſo that the Point Z ſhall never be reach'd. Again, the ſaid Sum will not be leſs, becauſe I have above ſhew'd, A Z, B Z, C Z, to be continually proportional; and in the ſame Manner the ſame Thing is ſhew'd of the reſt E Z, F Z, *&c.* Seeing therefore by transferring the Proportionals Q C, E R, F S, *&c.* into C E, E F, F I, the Remainders E Z, F Z, I Z, *&c.* are always continually proportional, as we have already ſhewed; we ſhall at the laſt come unto a Remainder leſs than any given one; and therefore the Sum of the Proportionals ſhall exceed every Quantity that is leſs than AZ; from whence itſelf cannot be leſs than AZ. Seeing therefore it is neither greater nor leſs than AZ, it ſhall be equal to it. *Q. E. D.*

Theorem.

THE Difference of the firſt Terms, the firſt Term, and the whole Sum of the infinite Proportionals, are continually proportional.

Fig. 19.

In the upper Figure let OX be drawn parallel to AZ. Therefore LX ſhall be the Difference of the firſt Term AL or AB, and of the ſecond BO, or BC. Becauſe XO is parallel to AZ; LX ſhall be to XO, as [u] LA is to AZ. But XO is AB, and LA likewiſe is AB. Therefore the Difference LX is to the firſt Term AB, as AB the firſt Term is to AZ the whole Sum. *Q. E. D.*

[u] *Per Corol.* 1. *Prop.* 4. *l.* 6.

Fig. 20.

The ſame Thing may be demonſtrated univerſally and very briefly in every kind of Quantity, thus: Let there be any continually Proportionals whatſoever (as well Numbers, as other Quantities) AZ, BZ, CZ, *&c.* and let them all be transferr'd upon the firſt AZ. Therefore AB, BC, CE, EF, *&c.* will be the Differences of the Proportionals; which together with the laſt Quantity IZ are equal to the firſt AZ. Now becauſe, if Proportionals be continued infinitely, the laſt Quantity vaniſheth away, it is manifeſt that the Differences of the infinite Proportionals are equal to the firſt AZ. Then becauſe AZ is to BZ, as BZ is to CZ, and ſo on. By dividing, AB will be to BZ, as

BC to CZ; and by converting, as AB, the firſt Difference, is to AZ, the firſt Quantity: ſo BC, the ſecond Difference, is to BZ, the ſecond Quantity, and ſo forwards. Therefore as AB, the firſt Difference, is to AZ the firſt Quantity: So all the Differences (that is, as I have already ſhew'd, the firſt Quantity AZ) are to all the Quantities, that is, to the whole Sum of the infinite Quantities. *Q. E. D.*

PROP. XII. Problem.

Fig. 21. *THREE right Lines being given (AB, BC, AF) to find a fourth Proportional.*

Let the two right Lines be diſpoſed, as the Figure ſhews, and draw the right Line BF, to which let the infinite Line CZ be made parallel. Let AF produc'd to L meet CZ.

I ſay, AB is to BC, as AF to FL, as is manifeſt from *Prop.* 2. of this Book. Therefore FL is the fourth Proportional ſought.

Scholium.

OUR Countryman *Bettin*, in his Treaſury of Mathematical Philoſophy, doth handſomely, from 35. l. 3 and 14 of this, which depends not upon the preſent Propoſition, find out a fourth Proportional, three being given, and a third too being given, after this manner:

Fig. 22. If three right Lines be given, let the ſecond CB and the third BD be join'd right to one another, ſo as to make one right Line, and let the firſt BA touch them in the Point B in what Angle you will. Through the Points

C, A, D, deſcribe a Circle[w], which let A B the firſt Line meet in the Point Z. B Z is a fourth Proportional.

[w] *Per* 5. *l.* 4.

For ſeeing the Rectangles A B Z, C B D, are equal[x], A B will be to B C, as B D to B Z, by the 14th of this Book, which, as was ſaid, depends not upon this.

[x] *Per* 35. *l.* 3.

If there be given two right Lines A B, B C; let B D equal to B C be join'd to B C, ſo as to make one ſtrait Line. Then let the firſt A B touch B C in B in any Angle. Then the reſt is as before, and B Z will be the third Proportional ſought.

Fig. 23.

The Demonſtration is the ſame; for ſeeing the Rectangles A B Z, C B D, are equal, A B will be to B C, as B D (that is, B C) is to B Z.

P R O P. XIII. Problem.

TWO right Lines given (A C, C B) to find a mean Proportional.

Fig. 24.

Let the whole compound Line A B be biſected in O, and from the Center O a Circle be deſcribed through A and B; from C erect a Perpendicular C F, meeting the Circumference in F.

I ſay, A C is to C F, as C F is to C B.

For let A F, B F be drawn; the Triangle [y] A F B is right angled, and from the right Angle there is drawn the Perpendicular F C to the Baſe. Therefore A C is to C F, as [z] C F is to C B.

[y] *Per* 31. *l.* 3.

[z] *Per Corol.* 1. *p.* 8. *l.* 6.

Corollary.

HENCE it is manifeſt, that if from any Point of the Circumference (as F) there be drawn a Perpendicular to the Diameter (F C), this Perpendicular is a mean Proportional betwixt the Segments of the Diameter (A C, C B).

Scholium.

THIS Place requires, that we ſhould ſay ſomething briefly concerning the finding out of two mean Proportionals betwixt two given Lines. All the Geometricians of *Greece*, at *Plato*'s Suggeſtion, ſet themſelves with all their Might to the Solution of this Problem. Divers moſt ſubtle Ways of Practice are recited by *Eutocius* in his Commentary on *Archimedes*; as thoſe of *Plato*, *Architas* the *Tarentine*, *Menæchmus*, *Eratoſthenes*, *Philo Byzantius*, *Hero*, *Apollonius* of *Perga*, *Nicomedes*, *Diocles*, *Sporus*, *Pappus*; to whom the later Times have added *Verner*, *Gregory* of St. *Vincent*, *Renatus Carteſius*. Out of all theſe we ſhall ſelect Three more eaſy than the reſt.

PLATO's *Method.*

Fig. 25. IT is requir'd to find out two Means betwixt the given Lines A B, B C.

Let A B, B C be ſet at a right Angle, and be produc'd infinitely towards X and Z. Then let two Squares (ſo our *Claudius Richards* hath it; for *Plato* himſelf made uſe of one Square only, but which had inſerted into its Side [a] D E, a Rule moveable along D E; let two Squares, I ſay)

[a] *See* Fig. 26.

I ſay) be taken, and the Angle D of one Square be applied to the right Line B X, in ſuch certain wiſe, that one Side may alſo paſs through A; and to the Point E, in which the other Side cuts the right Line B Z, let a ſecond Square be applied, which will paſs through C. I ſay, that B D, B E are two Means betwixt the given Lines A B, B C; that is, as A B is to B D, ſo is B D to B E, and B E to B C.

The Demonſtration is manifeſt from *Coroll.* 1. *Prop.* 8. *l.* 6. for A D E is a right-angled Triangle, and from the right Angle to the Baſe there falls the Perpendicular D B. Therefore by the ſaid Corollary, as A B is to B D, ſo is B D to B E; and for the ſame Cauſe, as B D to B E, ſo is B E to B C. Therefore betwixt the given right Lines A B, B C, there are found two mean Proportionals B D, B E. Which was the thing to be done. This Manner of ſolving the Problem is the eaſieſt of all to be underſtood.

The Method of PHYLO *the* Byzantine.

LET the two given right Lines AB, B C, be ſet together at a right Angle; then let the Rectangle A B C D be perfected, and let D A, D C be produc'd infinitely, and let the Diameters B D, A C be drawn, cutting each other in E. From the Center E through B let a Circle be drawn, which, becauſe A B C is a right Angle[b], will paſs through A and C. Then let a Rule be applied to the Point B, ſo that the intercepted right Lines B G, O F, may be equal. I ſay, that A F, G C, are two mean Proportionals betwixt the given A B, B C; that is, as A B is to A F, ſo is A F to G C, and G C to C B.

Fig. 27.

[b] *Per* 31 *l.* 3.

Demonſt. Becauſe GB, OF [c] are equal, OG, BF will be alſo equal. Therefore the Rectangles OGB, BFO, that is, the [d] Rectangles DGC, DFA, are equal. Therefore as GD is to DF, ſo [e] reciprocally AF is to GC; but GD is to DF [f], as BA to AF. Therefore as BA is to AF, ſo AF is to GC. Again, becauſe I have already ſhew'd that AF is to GC, as BA is to AF; and ſince BA is to AF, as GD is to DF; that is, as GC is to CB, AF will alſo be to GC, as GC is to CB. Therefore all four, BA, AF, GC, CB, are continually proportional; and therefore betwixt the given Lines AB, BC, two Means have been found. *Q. E. I.*

Theſe two Methods of Solution, although they be ingenious and eaſy enough; yet, becauſe a due Application of a Square and Rule is not made but by trying, they are not Geometrical.

The Method of CARTES.

Fig. 28. LET an Inſtrument of ſuch Sort be provided, that two Rules may be open'd and ſhut about Y. Let there be inſerted into theſe divers Squares connected together betwixt themſelves in the Points B, C, D, E, F, G, in ſuch ſort, that in the mean while that the Rules YX and YZ are open'd, the Square BC may impel the Square CD in the Rule YZ, and the Square CD may impel the Square DE in the Rule YX, and the Square DE may impel EF, and EF impel or force forward FG, and ſo on: But ſo that while the Rules YX and YZ are ſhut, all the Points B, C, D, E, F, G, tend to fall upon one and the ſame Point A. By this Inſtrument not only two, but alſo four and ſix,

[c] By the Conſtruction.

[d] *Per Corol.* 1. *p.* 36. *l.* 3.

[e] *Per* 14. *l.* 6.

[f] *Per Corol.* 1. *p.* 4. *l.* 6.

ſix, yea, as many Means as you will, betwixt two given right Lines may be found. Which thing can be obtain'd neither by the Sections of a Cone, nor by any Methods found out by the abovesaid Authors.

For two Means three Squares are requir'd, for four Means five Squares, and ſo on.

Let the leſſer of the given right Lines be transferr'd upon the Rule Y X, and let it be Y B; the greater upon the Rule Y Z, and let it be Y E. Let the firſt Square be applied to the Point B, and be fixed there; and let the Rules be open'd, until the Side of the third Square paſſeth through E. I ſay, that Y C, Y D, are two Means betwixt the given Y B, Y E; that is, that Y B is to Y C, as Y C is to Y D, and Y D to Y E.

The Demonſtration appears out of *Coroll.* 2. *Pr.* 8. *L.* 6. For from the Nature of the Inſtrument, in the Triangle Y C D, the Angle at C is a right one, and from it C B falls perpendicular upon the Baſe Y D. Therefore by the ſaid Corollary, as Y B is to Y C, ſo is Y C to Y D. Again, becauſe in the Triangle Y D E, the Angle at D is a right one, and from it there falls the Perpendicular D C upon the Baſe Y E, as Y C is to Y D, ſo is Y D to Y E. Therefore Y B, Y C, Y D, Y E are four continual Proportionals. Betwixt the given Line therefore Y B, Y E, there have been found two mean Proportionals Y C, Y D. *Q. E. I.*

If betwixt the given ones Y B, Y G, there be required four Means, open the Rules, until the Side of the fifth Rule F G paſſeth through G. There will be Y C, Y D, Y E, Y F, four Means betwixt Y B, Y G. The Demonſtration is manifeſt from the ſaid Corollary.

This Way, although the Inſtrument is more operoſe than *Plato's*, is in very Deed an excellent one; both becauſe it doth nothing by bare Trial, and becauſe it extends itſelf unto four and ſix, and as many Means you will.

The Deliacal Problem, to wit, the Duplication of the Cube is performed by two Means, and all Bodies whatſoever are increas'd or diminiſh'd in a given Proportion [g] by the ſame; like as the ſame thing is perform'd in plain Figures [h] by one Mean. *Hippocrates* firſt open'd this Way, which, as the ſingular and only one, all Geometricians that have follow'd him have embrac'd.

[g] See *ſchol.* *p.* 18. *l.* 12.
[h] *Corol.* 3. *pr.* 20. *l* 6.

PROP. XIV. Theorem.

Fig. 29, 30.

Equal Parallelograms (X, Z) which have one Angle (C) equal to one (O), have their Sides alſo, which are about the equal Angles, reciprocal; (that is, AC is to CB, as FO is to OL):

And, if they have the Sides thus reciprocal, the Parallelograms are equal.

Part I. Let IL and SB being produc'd meet together in Q. The Parallelogram X is to the Parallelogram R, as AC is to CB [i]; and Z is to R [k], as FO to OL. But, becauſe by the Hypotheſis X and Z are equal, X is to R as Z is to R. Therefore alſo AC is to CB, as FO is to OL. *Q. E. D.*

[i] *Per* 1. *l.* 6.
[k] By the ſame.

Part II. As AC is to CB, ſo X is to R [l]: And as FO is to OL, ſo is Z to R [l]. But already by the Hypotheſis AC is to CB, as FO to OL. Therefore X is to R, as Z is to R. Therefore X and R are equal. *Q. E. D.*

[l] By the ſame.

[Coroll. *On this depends the Demonſtration of the inverſe Rule of Proportion. For in it there is always ſome Rectangle given, as X; and one Side of another equal Rectangle, as CB; and the other Side is ſought. As therefore AC the firſt Side of the given Rectangle is to CB, the given Side of the other Rectangle; ſo reciprocally FC the ſought Side is to CL the ſecond Side of the given Rectangle. The Rectangle therefore CB × FC is equal to the Rectangle AC × CL: And, the latter Rectangle given being divided by the given Side of the former CB, the Quotient will give the ſought Side FC.* Q. E. I.]

PROP. XV. Theorem.

Equal Triangles (ACL, FCB) which have one Angle (C) equal to one (O) have alſo their Sides about the equal Angles reciprocal (that is, AC is to CB, as FO to OL). Fig. 31, 32.

And, if they have their Sides thus reciprocal, the Triangles are equal.

Let the right Line LB be drawn; the reſt of the Demonſtration is the ſame as that of the foregoing.

Corollary.

As well Parallelograms as Triangles, which have their Baſes and Altitudes reciprocal, are equal: And ſo converſely.

It is manifeſt from the two foregoing Propoſitions.

PROP. XVI. Theorem.

Fig. 33.

IF four right Lines (AB, FI; IL, BC) be proportional, (that is, if AB be to FI as IL is to BC) the Rectangle (X) under the Extremes (AB, BC) is equal to the Rectangle (Z) under the Means (FI, IL).

And, if a Rectangle under the Extremes be equal to a Rectangle under the Means, thoſe four right Lines will be proportional.

Part I. In the Rectangles X and Z, about the right, and therefore equal Angles B, I, by the Hypotheſis AB is to FI as reciprocally IL to CB. Therefore X and Z [m] are equal. *Q. E. D.*

Part II. Becauſe X and Z are now ſuppos'd equal, therefore [n] about the equal Angles B and I, AB is to FI as reciprocally IL to BC. *Q. E. D.*

[Coroll. (1.) *Hence it is eaſy to apply the given Rectangle Z* [o] *to the given right Line AB; to wit, by making AB : FI : : IL : BC. For BC is the Rectangle Z applied to the given right Line AB.*]

[Coroll. (2.) *Upon this Propoſition depends the Demonſtration of the direct Rule of Proportion. For in it there is always given ſome Rectangle, as CL: And another like Rectangle is ſought, one Side whereof is alſo given. It will therefore be, as BC, the firſt Side of the Rectangle given, is to EO the Side of the Rectangle ſought; ſo directly CE, the ſecond Side of the Rectangle given, is to OA the other ſought Side. Therefore the Rectangle*

[m] *Per* 14. *l.* 6.

[n] By the ſame.

[o] *Per* 12. *l.* 6.

angle $CE \times EO$ *is equal to the Rectangle* $BC \times OA$. *And the Rectangle* $CE \times EO$, *being divided by* BC *the Quotient, will give* OA *the other Side which was ſought.* Q. E. I.]

PROP. XVII. Theorem.

Fig. 34.

IF three right Lines (AB, FL, BC) be proportional, the Rectangle under the Extremes (AB, BC) ſhall be equal to the Square of the Mean (FL).

And, if the Rectangle under the Extremes be equal to the Square of the Mean, thoſe three right Lines are proportional.

Part I. Let O be taken equal to the Mean FL. Becauſe therefore by the Hypotheſis AB is to FL as FL to BC, and O is equal to FL; AB will alſo be to FL, as O is to BC. Therefore [p] the Rectangle under the Extremes AB, BC, is equal to the Rectangle under the Means FL and O, that is, is equal to the Square of FL.

Part II. This is demonſtrated in like manner from the ſecond Part of the foregoing.

Corollary.

Fig. 24.

FROM this, taken together with the 13th, it is manifeſt, that, if in a Circle FC be perpendicular to the Diameter, the Rectangle ACB is equal to the Square of FC.

[(2.) *If* $A \times B$ *be equal to the Square of* C; *then* $A : C :: C : B$.

(3.) *If* $A : C :: C : B$; *and* Cq *be divided by* A, *the Quotient* [q] *will be* B.

[p] By the foregoing.

[q] *Per Corol.* 2. *p.* 16. *l.* 6.

PROP. XVIII. Problem.

Fig. 35.

U*PON a given right Line (RS) to deſcribe a Polygon like, and in like manner poſited to a given one (BQ).*

Reſolve the given Polygon BQ into Triangles. Upon the given right Line RS make the Angles[r] R, O, equal to the Angles B, A. The Sides then will meet together in X. Upon XS make the Angles V, I, equal to the Angles T, C. The Sides will then meet together in Z. I ſay the Thing is done.

For becauſe the Angles R, O, are equal to the Angles B, A, the Angles E, K muſt alſo be equal (*per Corol.* 9. *pr.* 32. *l.* 1.); and becauſe alſo by the Conſtruction V is equal to T, the whole EV muſt be equal to the whole KT. In like manner becauſe O, I, are equal to A, C, reſpectively, the whole Angles OI, AC, muſt be equal. And, becauſe V and I alſo are equal to T and C by the Conſtruction, Z and Q likewiſe muſt be equal (*per Corol.* 9. *pr.* 32. *l.* 1.) to T and C. Therefore the Polygons RZ, BQ, are mutually equiangular. It remains, that we ſhew that their Sides alſo are proportional. RS is to BF[s] as SX to FL; and again, SX is to FL[t], as SZ to FQ. Therefore *ex æquo* RS is to SZ, as BF to FQ, *&c.*

Coroll. *Hence is derived the Method of making Maps or Charts, whether Geographical or Chorographical, or thoſe which Surveyors of Land make; and of framing Ichnographical Delineations of Fields, Buildings, Countries. For they are nothing elſe but the Reduction of great Figures*

[r] *Per* 23. *l.* 1.

[s] *Per* 4. *l.* 6.

[t] By the ſame.

unto like Figures which are of a ſmall Compaſs, which is performed by the Means of this Propoſition.

PROP. XIX. Theorem.

THE *Proportion of like Triangles (X, Z) is duplicate of the Proportion of their Sides (AC, FI) which are ſubtended to the equal Angles.* Fig. 36, 37.

That is, if it be made [u] as A C is to F I, ſo is FI to a third AQ; the Triangle X is to Triangle Z, as A C the firſt is to the third Proportional A Q. See *Defin.* 10. 5.

Becauſe the Triangles X, Z are like, B A, will be to L I [w] as A C is to I F. But by the Conſtruction, as A C is to I F, ſo is I F to A Q. Therefore alſo B A is to L I [x], as I F to A Q. Therefore in the Triangles Q B A and Z, the Sides about the Angles A, I, which by the Definition of like Triangles are equal, are reciprocal. Therefore Q B A and Z are equal [y]. But the Triangle X is to Q B A, as the Baſe A C to the Baſe A Q. Therefore X is to Z as A C to A Q. *Q. E. D.*

[u] *Per* 11. *l.* 6.

[w] *Per* 4. *l.* 6.

[x] *Per* 15. *l.* 6.

[y] *Per* 1. *l.* 6.

Coroll. *Hence is their Error to be corrected, who think that like Figures are in the ſame Proportion to one another, that their Sides are. For if of two, not only like Triangles, but alſo Squares, Pentagons, Hexagons,* &c. *yea, and Circles alſo, the Sides or Diameters be betwixt themſelves as* 2 *to* 1, *the Figures or Areas themſelves are as* 4 *to* 1: *If the Sides be betwixt themſelves as* 3 *to* 1, *the Figures themſelves or Areas are as* 9 *to* 1; *to wit, in a duplicate Proportion of thoſe Sides.*

PROP.

PROP. XX. Theorem.

Fig. 38.

LIKE Polygons (ABCDE, FGHIK) are divided, (1.) *into like Triangles (P, S, and Q, T, and R, V) in Number equal:* (2.) *And proportional to the Wholes: And* (3.) *the Proportion of the Polygons is duplicate to that of the Sides, (AB, FG) which are betwixt the equal Angles (B, G, and BAE, GFK).*

Part I. Becauſe the Polygons are like, they are mutually (*per Defin.* 1. *l.* 6.) equiangular, and their Angles equal B A E to G F K, and B to G, and B C D to G H I, and C D E to H I K, and E to K. Becauſe therefore A B is to B C[z] as F G to G H, and the Angles B and G are equal, the Triangles P, S,[a] are like. In like manner it will be demonſtrated that R and V are like. Then becauſe the Wholes B C D, G H I, and the ſubducted ones B C A, G H F are equal, the remaining ones alſo, A C D, F H I are equal. In the ſame manner I might ſhew that A D C, F I H are equal. Therefore (*per Corol.* 9. *pr.* 32. *l.* 1.) the third C A D is equal to the third H F I. Where alſo[b] the Triangles Q and T are like. The firſt Part thereof is manifeſt.

[z] By the ſame.

[a] *Per* 6. *l.* 6.

[b] *Per* 4. *l.* 6.

Part II. Becauſe P and S are like, the Proportion of P to S is duplicate to that of[c] C A to H F. But for the ſame Cauſe alſo the Proportion of Q to T is duplicate to the Proportion of C A to H F. Therefore P is to S as Q to T. In the ſame manner I will ſhew that as Q is to T, ſo R is to V. Therefore as one Antecedent

[c] By the foregoing.

Antecedent P is to one Conſequent S, ſo all the Antecedents P, Q, R, taken together, are to all the Conſequents, S, T, V, taken together, that is, ſo is Polygon to Polygon. Which was the other Part.

Part III. The Proportion of P to S is duplicate[d] to that of A B to F G. But the Proportion of Polygon to Polygon is the ſame with the Proportion of P to S, as I have already ſhew'd. Therefore alſo the Proportion of Polygon to Polygon is duplicate to the Proportion of A B to G F. Which was the third Part.

[d] By the foregoing.

Corollaries.

1. ALL ordinate or regular Figures, as Squares, equilateral Triangles, Pentagons, *&c.* are betwixt themſelves in the duplicate Proportion of the Sides. For all regular Figures are like, as is manifeſt from *Defin.* 1. 6.

2. If, in any like Figures whatſoever, the Sides A B, F G, which are placed betwixt equal Angles be known, the Proportion of the Figures is alſo known. As, for Example, let A B be of two Feet, and F G of ſix Feet, and as 2 is to 6, ſo let 6 be to ſome other Number; to wit, 18. The leſſer Figure is to the greater as 2 is to 18, or as 1 is to 9. Now a third proportional Number is found, if (*per Corol.* 3. *pr.* 17. *l.* 6.) the ſecond of the given ones be multiplied by itſelf, and the Product divided by the firſt. Fig. 38.

3. From the ſame Propoſition is drawn the excellent Method of increaſing or diminiſhing any rectilineal Figure in a given Proportion. As if I would make a Pentagon, whoſe Side Fig. 39.

is A B fivefold of another. Find a mean Proportional B X[e] betwixt the Terms of the Proportion given, A B, B C; upon this frame[f] a Pentagon like to the given one. This ſhall be quintuple of the given one.

[e] *Per* 13. *l.* 6.
[f] *Per* 18. *l.* 6.

For by the 20th the Pentagon A B is to B X, which is like to it, as A B the firſt is to B C the third Proportional.

Moreover, ſeeing the Proportion of Circles alſo is duplicate to the Proportion of their Diameters, as will be ſhew'd, *p.* 2. *l.* 12. This Practice belongs likewiſe to Circles.

Fig. 41. [Schol. *Seeing the Proportion of the Squares E, K, is duplicate of the Proportion of their Sides, O R, S V; from thence the duplicate Proportion of the Sides, O R, S V is wont commonly to be expreſs'd by the Proportion of O R q to S V q.*]

PROP. XXI. Theorem.

Fig. 40. F*IGURES* (*A, B*) *which are like to the ſame* (*C*) *are alſo like betwixt themſelves.*

This is manifeſt from *Defin.* 1. *l.* 6. and from *Axiom* 1. *l.* 1.

PROP. XXII. Theorem.

Fig. 40, 41. I*F four or more right Lines* (*F I, L Q, and O R, S V*) *be proportional; like Figures, and in like Sort deſcribed by them* (*A, B, and E, K*) *muſt alſo be proportional. And converſly.*

The Demonſtration of the firſt Part is manifeſt. For becauſe the Proportions of A to B and E to K are duplicate to the Proportions of F I to L Q, and O R to S V, which are by Hypotheſis equal; themſelves alſo muſt be equal.

The ſecond Part is manifeſt alſo.

[Coroll. *If the right Line A B be cut in any manner in C; the Rectangle contain'd under the Parts A C, C B, is a mean Proportional betwixt their Squares. Likewiſe the Rectangle contain'd under the Whole A B, and one Part A C or C B, is a mean Proportional betwixt the Square of the Whole A B, and the Square of the ſaid Part A C or C B. For* (per Coroll. 1. p. 8. l. 6.) *it is manifeſt that A C : C F :: C F : C B. Therefore A C Square : C F Square :: C F Square : C B Square. That is*[g], *A C Square : Rectangle A C B :: Rectangle A C B : C B Square.* Q. E. D. Fig. 24.

Moreover, (per Coroll. 2. p. 8. l. 6.) *B A : A F :: A F : A C. Therefore B A q : A F q :: A F q : A C q. That is*[h], *B A q : B A C Rectangle :: B A C Rectangle : A C q. In the ſame manner A B q : A B C :: A B C : B C q.* Q. E. D.]

[g] *Per* 17. *l.* 6.

[h] *Per.* 17. *l.* 6.

PROP. XXIII. Theorem.

E*Quiangled Parallelograms (X, Z) have betwixt themſelves a Proportion that is compounded of the Proportions of their Sides (A C to C B, and L C to C F).* Fig. 42.

That is, if you make C B to be to O, as L C to C F, X is to Z, as A C is to O.

Let I L, S B, meet together in Q. The Parallelogram X[i] is to the Parallelogram R,

[i] *Per* 1. *l.* 6.

[k] By the ſame. as AC is to CB; and R is[k] to Z as LC is to CF, that is, as CB is to O. Therefore *ex æquo* X is to Z, as AC is to O. *Q. E. D.*

Corollaries.

Fig. 42. FROM hence, and from 34. *l.* 1, it is manifeſt,

1. That Triangles which have one Angle (at C) equal, have that Proportion betwixt themſelves, which is compounded of the Proportions of the right Lines AC to CB, and LC to CF. Which Lines contain the equal Angle.

2. That Rectangles, and conſequently all Parallelograms whatſoever, have betwixt themſelves the Proportion, which is compounded of the Proportions of the Baſe to the Baſe, and the Height to the Height. And in the ſame manner we reaſon about Triangles.

Fig. 42. 3. Hence the Proportion of Triangles and Parallelograms may be readily learned. Let X and Z be the Parallelograms, and their Baſes AC, CB, and CL, CF be their Heights. Let it be made[l] as the Altitude CL is to the Altitude CF, ſo is one of the Baſes CB to O. The Parallelogram X is to the Parallelogram Z, as AC to O.

[l] *Per* 12. *l.* 6.

PROP. XXIV. Theorem.

Fig. 43. *IN every Parallelogram (as SF) the Parallelograms which are about the Diameter (AB), to wit (CL, OL) are both like to the whole Parallelogram, and to each other.*

By

ELEVENTH BOOK

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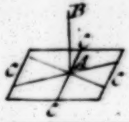

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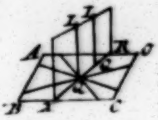

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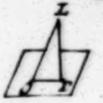

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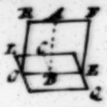

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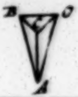

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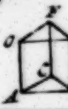

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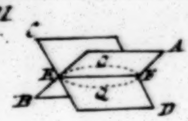

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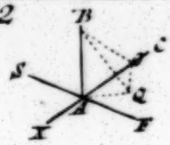

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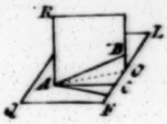

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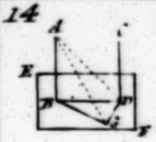

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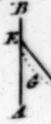

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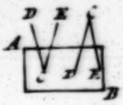

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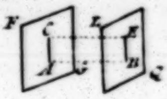

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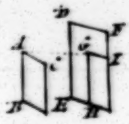

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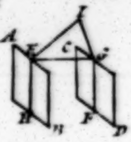

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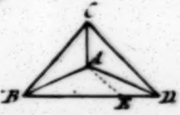

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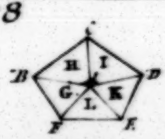

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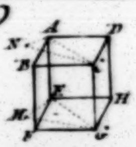

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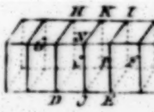

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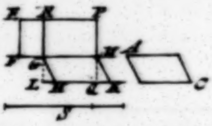

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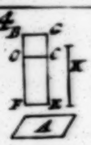

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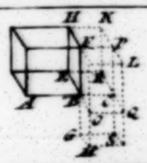

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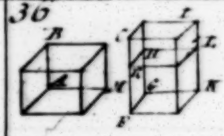

TWELFTH BOOK

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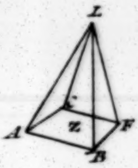

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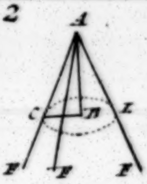

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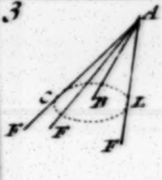

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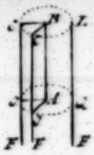

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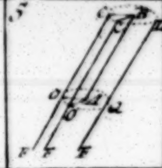

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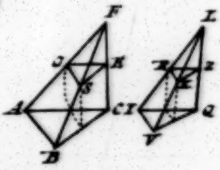

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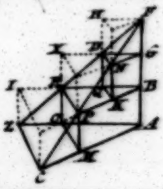

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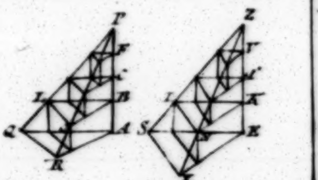

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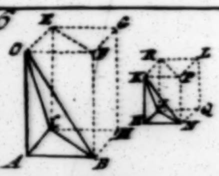

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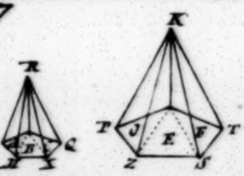

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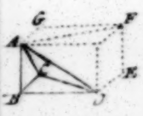

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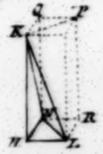
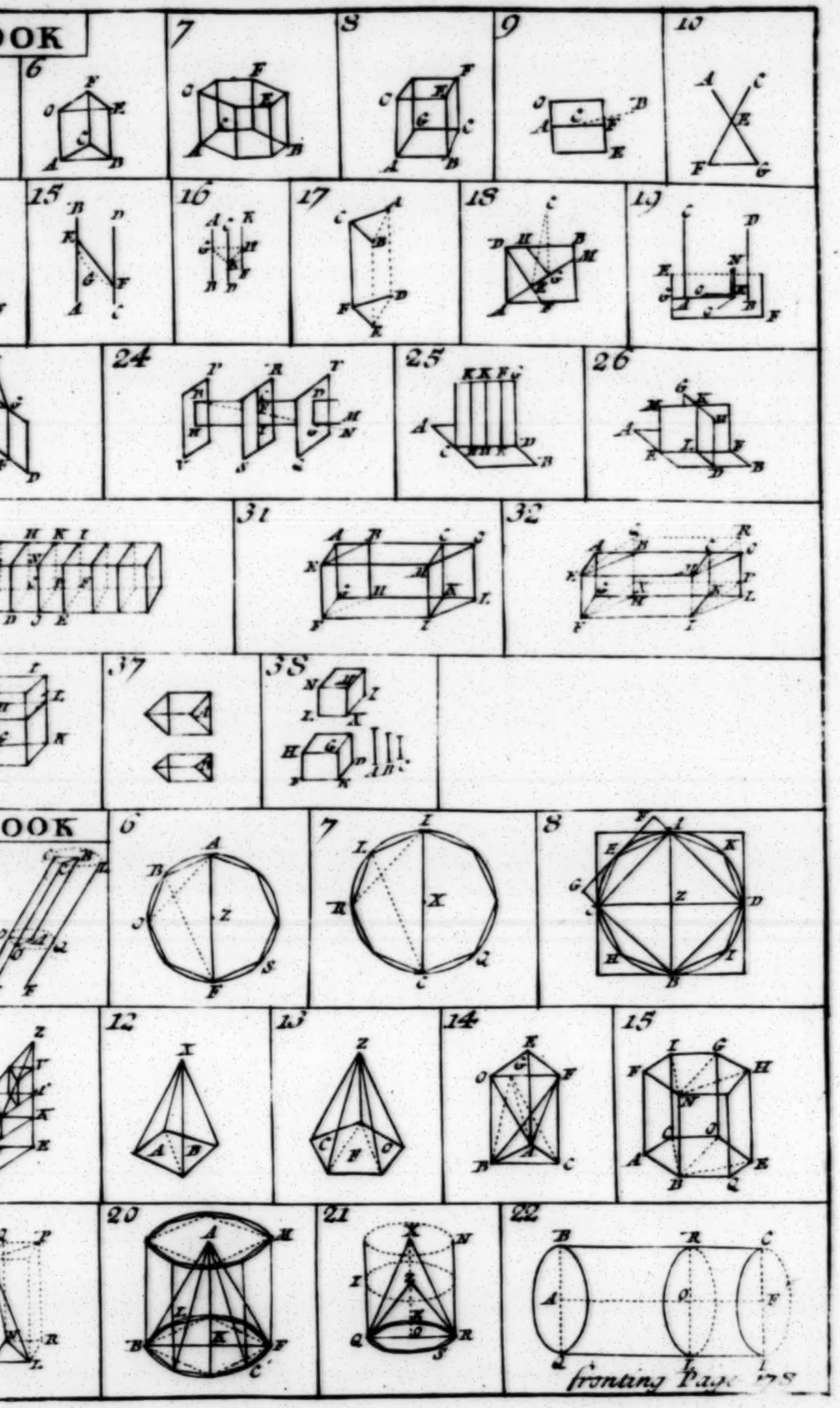


I. Senex ſculp.



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By 27. 1. the Angles C, S, and L, F, are equal. By the ſame E is equal to I, that is, by the ſame, equal to A itſelf; but B is common both to the Whole SF, and the Part CL. Therefore the Whole SF, and the Part CL, are equiangular. It remains to be ſhew'd that they have the Sides oppoſite to the equal Angles proportional.

Becauſe in the Triangles BCE, BSA, CE is parallel to SA, BC (by *Corol.* 1. *pr.* 4. *l.* 6.) will be to CE, as BS to SA: And CE will be to EB (by the ſame *Corol.*) as SA to AB. But becauſe in the Triangles ELB, AFB alſo, EL is parallel to AF, EB (by the ſame *Corol.*) will be to EL, as AB to AF. Therefore *ex æquo* CE is to EL, as SA to AF. Therefore (by *Defin.* 1. *l.* 6.) CL and the Whole CF are like. In the ſame manner, I might ſhew OI to be like to the Whole SF. Therefore (*per* 21. *l.* 6.) CL and OI are alſo like betwixt themſelves. *Q. E. D.*

PROP. XXV. Problem.

TO change a given Polygon (A) into another like to a given one (B). Fig. 46.

Or to make a Polygon equal to a given one (A) and like to another given one (B).

Upon CF the Side the Polygon B, a like one to which is required, (by 45. *l.* 1.) make a Rectangle Q equal to B. Then upon FS (by the ſame *Prop.*) make a Rectangle R equal to A. It is manifeſt that CF and BI do make one right Line. Betwixt CF and FI find a mean Proportional FL[m]. Upon this (*p.* 18. *l.* 6.)

[m] *Per* 13. *l.* 6.

l. 6.) make a Polygon like to the given one B. This muſt alſo be equal to the given one A.

For, ſeeing by the Conſtruction C F, F L, F I, are three Proportionals, the Polygon B is to the Polygon like to it which is made upon F L, as C F is to F I (*per* 20. *l.* 6. and *Def.* 10. *l.* 5.); that is, (*per.* 1. *l.* 6.) as Q is to R. Therefore alſo by changing, as the Polygon B is to Q, ſo is the Polygon F L to R. But by the Conſtruction the Polygon B is equal to Q. Thereſore alſo the Polygon upon F L, which is like to B, is equal to R; that is, by the Conſtruction to the given A. That therefore is done which was required.

PROP. XXVI. Theorem.

Fig. 44. *LIKE Parallelograms* (*B D, F N*) *having a common Angle* (*A*) *are about the ſame Diameter.*

Draw the right Lines A E, C E. If you deny that A E C is a common Diameter to the Parallelograms B D and F N; let another right Line A G C, which cuts F E in G, be the Diameter of B D, and draw the Parallel G H. The Parallelograms F H, B D will be therefore about the common Diameter A G C, and conſequently (by 24. *l.* 6) will be like. Therefore (*per Defin.* 1. *l.* 6.) as B A to A D, ſo is F A to A H. But alſo, as B A to A D, ſo is F A to A N, ſeeing B D, F N are like by the Hypotheſis. Therefore F A is to A H, as the ſame F A is to A N. Which is abſurd.

PROP.

PROP. XXVII, XXVIII, XXIX.

THESE cauſe Trouble to, and perplex Beginners, and are ſcarce of any Uſe.

PROP. XXX. Problem.

TO cut a given right Line (AB), ſo that the Whole (AB) ſhall be to one Segment (AC) as the ſame Segment is to the Remainder (CB). Fig. 45.

That is, as Geometricians ſpeak, to cut a Line in extreme and mean Proportion.

By 11. *l.* 2. ſo cut AB in C, that the Rectangle under AB, CB may be equal to the Square of AC. I ſay the Thing is done.

For, by the 17th of this Book, as AB is to AC, ſo is AC to CB.

The Force of this Section of a Line is admirable in the inſcribing and comparing regular Bodies.

PROP. XXXI. Theorem.

IF from the Sides of a rectangular Triangle (ACB) like Figures whatſoever be deſcrib'd, that which is oppoſ'd to the right Angle will be equal to the two others (L, R) taken together. Fig. 47.

Here *Prop.* 47. *l.* 1. is made univerſal.

From the right Angle C let the Perpendicular CO be let down. Becauſe (*per Coroll.* 2. *p.* 8. *l.* 6.) AB, BC, BO are three Proportionals,

tionals, F ſhall be to the Figure R, which is like to it, as AB the firſt, to BO the third Proportional, (to wit, by 20. *l.* 6. and *Def.* 10. *l.* 5.) Again, becauſe (by the aforeſaid Corollary) BA, AC, AO are three Proportionals, the Figure F ſhall (by the foreſaid *Prop.* and *Defin.*) be to L, which is like to it, as BA the firſt, to AO the third Proportional. Becauſe therefore F is to R as AB is to BO; and the ſame F is to L as AB to AO; F ſhall alſo be to R and L taken together, as AB is to BO, AO taken together. But AB is equal to the two BO, AO. Therefore alſo F ſhall be equal to the two R and L. *Q. E. D.*

Corollaries.

FROM this Propoſition we can eaſily find one rectilineal Figure, equal and like to any Number of rectilineal Figures whatſoever, by the ſame Method, whereby *Probl.* 1. *Schol. pr.* 47. *l.* 1. one Square is found equal to any Number of given Squares whatſoever. Only, in the Demonſtration, let 31. *l.* 6. be cited inſtead of 47. *l.* 1.

Coroll. (2.) *A Circle upon the Hypotenuſe of a rectangle Triangle is equal to two Circles deſcrib'd upon the Sides. For all Circles are like amongſt themſelves; and are to one another as the Squares of their Diameters, by the ſecond of the* 12*th Book.*

Fig. 54. Coroll. (3.) *From hence we may derive that Quadrature of Lunets (or little Moons) which* Hippocrates *of* Chios *firſt taught.*

For let ABC be a rectangle Triangle; and BAC a Semicircle to the Diameter BC: BNA a Semi-

a Semicircle deſcrib'd on the Diameter AB: AMC a Semicircle deſcrib'd upon the Diameter AC. Thus therefore the Semicircle BAC is equal to the Semicircles BNA and AMC together. If therefore you take away the two Spaces BA, AC, common on both Sides, there will be left the two Lunets BNA, AMC, bounded on both Sides with circular Lines equal to the rectilineal Triangle BAC. And if the Line BA be equal to the Line AC, and you let fall a Perpendicular unto the Hypotenuſe BC, the Triangle BAO will be equal to the Lunet BNA, and the Triangle COA equal to the Lunet CMA. Q. E. I.

PROP. XXXII.

THIS *is hardly of any Uſe, and hath nothing remarkable in it.*

PROP. XXXIII. Theorem.

Fig. 48.

I*N the ſame or equal Circles, the Angles whether at the Centres (as ABC, FOD) or at the Circumference (as ARC, FSD) have that Proportion betwixt themſelves, which the Arches (AKC, FGD) on which they ſtand have. Underſtand the ſame thing of Sectors.*

As for the Angles at the Centre and the Sectors, it will be demonſtrated altogether in the ſame manner, in which *Prop.* 1. of this Book it was demonſtrated, that Triangles of the ſame Height are as their Baſes. Only where *Prop.* 38. *l.* 1. is cited there, let *Prop.* 29. *l.* 3. be cited here.

And becauſe the Angles R and S at the Circumference are Halves of the Angles A B C, F O D, at the Centre, that which hath been demonſtrated of theſe will be manifeſt alſo of thoſe.

Coroll.

Fig. 49. 1. THE Angle (B A C) at the Centre, is to four right Angles, as the Arch B, on which it ſtands, is to the whole Circumference.

For, as B A C is to the right Angle B A F, ſo by this 33. the Arch B C is to the Quadrant B F. Therefore the Angle B A C is to four right Angles, as the Arch B C is to four Quadrants, that is, the whole Circumference.

2. The Arches I L, B C of unequal Circles, which do ſubtend equal Angles, whether at the Centre, as I A L and B A C, or at the Circumference, are like Arches.

For, the Arch I L is (by *Coroll.* 1.) to its Circumference, as the Angle I A L, that is, B A C is to four right Angles; and the Arch B C is to its Circumference (by the ſame *Corollary*) as the ſame Angle B A C is to four right ones. Therefore I L is to its Circumference, as B C is to its. Therefore (by *Defin.* 4. *l.* 6.) the Arches I L and B C are like.

3. Two Semidiameters (A B, A C) do take away from concentrical Circumferences like Arches I L, B C. This is manifeſt from *Coroll.* 2.

4. The Segments (B K C, I O L) which contain equal Angles (K, O) are like.

For, by *Coroll.* 2, the Arches B C, I L, and conſequently the Arches B K C, I O L are like.

THE

THE ELEMENTS OF EUCLID.

BOOK XI.

With us the SEVENTH.

TO the Six firſt Books *Euclid* ſubjoins the Elements of Numbers comprehended in the three following, the Seventh, Eighth, and Ninth; to which he alſo adjoins a Tenth, concerning incommenſurable Quantities. We paſs immediately from Plains to Solids, purpoſing to treat of Numbers ſeparately. Seeing it will, I ſuppoſe, be more commodious for Learners, if the Elements of Geometry be not interrupted by treating of any other Matter, but be had all together. Nevertheleſs, when we ſhall cite the

Propoſitions of this and the following Book, we ſhall not call theſe Books the Seventh and the Eighth, but the Eleventh and the Twelfth, leſt, if we ſhould depart from the every-where received Order of *Euclid*, the Citation of Propoſitions ſhould thereby be render'd more intricate.

This Book in a ſort contains two Parts. In the firſt are laid the Foundations on which the whole Doctrine of Solids, that is, of Bodies depends. In the other the Affections of Parallelopipeds are propounded.

This Eleventh Book of Elements ſets forth the firſt Principles of Solids. Nor can indeed the Properties of Bodies be known without it; and, if we ſet upon almoſt any Part of the Mathematicks without the Knowledge of Solids, we ſhall labour in vain, or be at leaſt at a great Loſs. For the ſpherical Doctrine of Theodoſius, *ſpherical Trigonometry alſo, a great Part of practical Geometry, Staticks and Geography depend upon it; and what things occur of any great Difficulty in the Art of Dialling, in the Conic Sections, Aſtronomy, Dioptricks or Opticks, do all become more eaſy, the Principles of Solids being once underſtood. So that thoſe who have delivered the Elements of Geometry, leaving out and ſetting aſide this and the following Book, are to be reckon'd to have delivered the ſame very imperfectly.*

DEFINITIONS.

1. A Solid or Body is that which hath Length, Breadth, and Thickneſs.

2. The Extreme of a Solid is a Surface.

3. The

Fig. 1. *l.* 11. 3. The right Line (A B) is to the Plain (C C) right or perpendicular, when it makes right Angles (B A C, B A C) with all the right Lines (C A) in the Plain (C C) by which it is touch'd.

Fig. 2. 4. A Plain is right or perpendicular to a Plain, when all the right Lines (L Q) which are drawn in one of the Plains perpendicular to the common Section (X R) are right or perpendicular to the other Plain (A B C O).

Fig. 3. 5. If the right Line (O L) ſtands upon a Plain not at right Angles, and from its higheſt Point (L) there be drawn to the Plain the Perpendicular (L P) and (O P) be join'd; the Angle (L O P) is ſaid to be the Inclination of the Line (O L) to the Plain.

Fig. 4. 6. If the Plain (R E) doth not ſtand perpendicularly upon the Plain (L Q), the Inclination of one to the other is the acute Angle (A B C), which is contain'd by the right Lines (A B and B C) which are drawn in both Plains perpendicular to the common Section (O E).

7. A Plain is ſaid to be alike inclin'd to a Plain, as is ſome other Plain to another, when the ſaid Angles of their Inclinations are equal.

8. Parallel Plains are thoſe which, being continued every way, are always diſtant from each other by equal Intervals.

9. Like ſolid rectilineal Figures are thoſe which are contain'd under like Plains, in Number equal.

Fig. 5. 10. A ſolid right-lin'd Angle is that which is contain'd under plain Angles more than two (B A C, C A O, O A B) which are not in the ſame Plain, meeting together in one Point.

11. Equal ſolid Angles are thoſe, which, being conceiv'd to be put each within the other, do agree or perfectly coincide.

Like as a plain Angle is a mutual Inclination of Lines, ſo a ſolid Angle is an Inclination of Surfaces. Concerning both therefore we muſt reaſon in the ſame manner.

Fig.6,7,8. 12. A Priſm is a ſolid Figure comprehended by Plains, amongſt which two oppoſite ones (OFE, ACB) are parallel, equal, and like.

Fig. 8. 13. A Parallelopiped is a Solid contain'd under quadrilateral Plains, of which the Oppoſites are parallel.

14. If ſix Plains in which the Oppoſites are parallel be Squares, the Solid contain'd by them will be a Cube.

PROPOSITION I. Theorem.

Fig. 9. *ONE Part (AC) of a right Line cannot be in a Plain (OE), and another Part (CB) out of it.*

It is clear of itſelf from the Definition of a Plain and a right Line. See *Defin.* 7 and 4. *l.* 1.

PROP. II. Theorem.

Fig. 10. *EVery Triangle is in one Plain. And two right Lines cutting each other, are in the ſame Plain.*

For if a Plain be applied to one of its Sides, and to the Point of meeting of the other two, it will be evident that the whole Triangle is in that Plain.

PROP. III. Theorem.

IF two Plains (AB, CD) cut each other, EF, their common Section, is a right Line. Fig. 11.

It is manifeſt from the Definition of a Plain. But we may demonſtrate it thus: If EF the common Section be not a right Line, let there be drawn in the Plain CD the right Line EOF, and in the Plain AB the right Line EQF. The two right Lines therefore EOF, EQF, will incloſe a Space. Which is abſurd.

PROP. IV. Theorem.

IF a right Line (BA) be perpendicular to two right Lines (CAX, FAS) which cut each other, it will alſo be perpendicular to the Plain which is drawn through them. Fig. 12.

If you deny it, let another right Line BQ be perpendicular to the Plain of the right Lines AC, AF. Join AQ, and to this in the Plain FAC draw the Perpendicular QO. This being produced will neceſſarily cut (as is gathered from *Schol. Prop.* 31. *l.* 1.) one of the right Lines CAX, FAS, or both, whereſoever the Point Q ſhall be. Therefore let it cut CAX in O, and let BO be join'd. Becauſe therefore the Angle BAO is by the Hypotheſis a right one;

The Square of BO ſhall be equal to $\left.\begin{array}{c}\text{BA Squ.}\\ + \\ \text{AO Squ.}\end{array}\right\}$ [a]

[a] *Per* 47. *l.* 1.

But becauſe B Q is ſuppos'd perpendicular to the Plain FAC, and conſequently (by *Defin.* 3. *l.* 11.) makes a right Angle with A Q;

B A Squ. is equal to B Q Squ.
+
A Q [b]

[b] *Per* 47. *l.* 1.

And becauſe the Angle A Q O is by the Conſtruction a right one;

A O Squ. is equal to O Q Squ.
+
A Q Squ. [c]

[c] By the ſame.

Therefore B O Squ. is equal to B Q Squ. +
O Q Squ. +
A Q Squ. twice taken.

Therefore B O Square is greater than the Squares of B Q and O Q; and (as is clear from *Prop.* 47. *l.* 1.) conſequently B Q O is not a right Angle. Therefore B Q is not perpendicular to the Plain (by *Defin.* 3. *l.* 11.) C A F. Therefore the Propoſition is manifeſt.

Scholium.

FROM its being ſuppos'd that B Q is perpendicular to the Plain FAC, it is directly demonſtrated that it is not perpendicular to that Plain; and conſequently, from the Denial of the Aſſertion of the Theorem, the ſame Aſſertion is directly proved. This Demonſtration, as to the Subſtance of it, is *John Cierman*'s.

PROP.

PROP. V. Theorem.

IF three right Lines (B A, C A, F A) be perpendicular to the ſame right Line (A R) at the ſame Point (A), thoſe three will be in one Plain. Fig. 13.

For, if it may be, let one of them B A be in another Plain (R O), which may cut L Q the Plain of the other two C A, F A, in the right Line A O. Becauſe by the Hypotheſis R A ſtands perpendicularly upon the two C A, F A, it will be perpendicular to the Plain L Q (by the foregoing.) Therefore R A makes a right Angle with A O (by *Defin.* 3. *l.* 11.) But alſo by the Hypotheſis R A B is a right Angle. Therefore the Angles R A B and R A O are equal. Which is abſurd.

PROP. VI. Theorem.

RIGHT Lines (A B, C D), which are perpendicular to the ſame Plain (C F), are parallel. Fig. 14.

It might be taken for granted as a thing of itſelf known; but we may demonſtrate it thus:

B D being join'd, make in the Plain F E the Line D G perpendicular to B D, and equal to B A; and let D A, G A, G B, be join'd. The right Lines B D, D G, are equal to B D [d] and B A; and the Angles B D G, [e] D B A, are right ones. Therefore (*per* 4. *l.* 1.) A D, B G, are equal. Therefore the Triangles A B G, G D A, are equilateral to each other, and conſequently

[d] By the Conſtruction.

[e] *Per Def.* 3. *l.* 11.

ſequently the Angles A B G, A D G, are equal. But A B G (by *Defin.* 3. *l.* 11.) is a right Angle. Wherefore A D G is alſo a right one. But B D G alſo by the Conſtruction, and C D G by *Defin.* 3, are right Angles. Therefore G D is perpendicular to the three Angles C D, A D, B D. Therefore C D is [f] in one Plain with A D and B D. But A B alſo is in one Plain (*per* 2. *l.* 11.) with A D and B D. Therefore A B, C D are in one Plain. Therefore ſeeing the Angles A B D, C D B (by *Defin.* 3. *l.* 11.) are right ones, A B, C D will (*per* 29. *l.* 1. and *Defin.* 36. *l.* 1.) be parallel Lines. *Q. E. D.*

[f] By the foregoing.

PROP. VII. Theorem.

Fig. 15.

A Right Line (E F) cutting right Lines (A B, C D) placed in the ſame Plain, is in one and the ſame Plain with them.

It might be taken for granted: But he that will may thus demonſtrate it:

Let another Plain cut the Plain of the right Lines A B, C D, in the Points E, F. If now E F is not in the Plain of A B, C D, it is not the common Section. Let E G F therefore be ſo. Therefore (*per* 3. *l.* 11.) E G F is a right Line; the two right Lines therefore E F, E G F, incloſe a Space. Which is abſurd.

Corollary.

HENCE it follows, that, if E F cut the Parallels A B, C D, it is in the ſame Plain with them. For (by *Defin.* 36. *l.* 1.) any two Parallels are in the ſame Plain.

PROP. VIII. Theorem.

IF of two Parallels (AB, CD) one (AB) be perpendicular to a Plain (EF), the other alſo (CD) will be perpendicular to the ſame Plain. Fig. 14.

It might be taken for granted. If the Demonſtration be requir'd, it is as follows :

BD, AD being drawn, in the Plain EF make GD perpendicular to BD. It will alſo (ſee the Demonſtration of Prop. 6. l. 11.) *be perpendicular to AD. Therefore* (per 4. l. 11.) *GD will be perpendicular to the Plain ABD, that is (by the foregoing* Coroll.) *to the Plain CBDA. Wherefore* (per Def. 3. l. 11.) *CDG is a right Angle. But the Angle CDB is alſo a right one (foraſmuch as with* **ABD**, *which* (per. Defin. 3. l. 11.) *is a right Angle, it maketh two right ones* (per 27. l. 1.)). *Therefore* (per 4. l. 11.) *CD is perpendicular to the Plain GDB or EF.* Q. E. D.]

PROP. IX. Theorem.

RIGHT Lines (AB, EF) which are parallel to the ſame right Line (CD) although they be not in the ſame Plain with it, are alſo parallel betwixt themſelves. Fig. 16.

Although it might be taken for granted, yet we will demonſtrate it thus :

In the Plain of the Parallels AB, CD, draw GK perpendicular to CD. Likewiſe in the Plain of the Parallels EF, CD, draw HK

perpen-

perpendicular to CD. Therefore [g] CK is perpendicular to the Plain GKH. Therefore, ſeeing AG, EH be parallel to CK, the ſame AG, EH [h] will be perpendicular to the Plain GKH. Therefore AG, EH [i] are parallel. *Q. E. D.*

[g] *Per* 4. *l.* 11.

[h] *Per* 8. *l.* 11.

[i] *Per* 6. *l.* 11.

PROP. X. Theorem.

Fig. 17. *If two right Lines (AC, BC) be parallel to two right ones (DF, EF), albeit they be not in the ſame Plain, they comprehend equal Angles (C and F.)*

Let CA, CB be made equal to FD, FE, and let DE, AB, DA, FC, EB be drawn. Seeing AC, FD are parallel and equal, AD alſo and CF will [k] be parallel and equal. In like manner I might ſhew BE, CF to be parallel and equal. Therefore AD, BE are alſo parallel [l] and equal, *(per Axiom* 1.) Therefore *(per* 33. *l.* 1.) AB, DE are equal. Seeing therefore the Triangles BAC, EDF are equilateral to each other, the Angles C and F [m] are equal. *Q. E. D.*

[k] *Per* 33. *l.* 1.

[l] By the foregoing.

[m] *Per* 8. *l.* 1.

PROP. XI. Problem.

Fig. 18. *To draw a Perpendicular to a given Plain (AB) from a Point given without it (C).*

The Conſtruction. In the Plain AB draw any right Line, as DF; unto which, from C, erect the Perpendicular CE. Then in the Plain AB through E draw AEM perpendicular to the ſame DF. Then to AM from C draw the Perpendicular CG. I ſay, that CG is perpendicular to the Plain.

Through G let HG be drawn parallel to DF. By the Conſtruction DE is perpendicular to CE and EM. Therefore DE is perpendicular to the Plain EM[n], as alſo is HG[o]. Therefore (by *Defin.* 3. *l.* 11.) CG is perpendicular to HG. But CG by the Conſtruction is alſo perpendicular to EM. Therefore [p] CG is perpendicular to the Plain AB. Which was the thing propos'd.

[n] *Per* 4. *l.* 11.
[o] *Per* 8. *l.* 11.
[p] *Per* 4. *l.* 11.

[Scholium. *In Practice thus: Let there be a Cord or Rule faſtened to the given Point A. And from the ſame let there be deſcrib'd by the End of it B in the Plain given the Circle BCFL. The Line AK, which connects the given Point and the Centre of the Circle, will be perpendicular to the given Plain.*] Fig. 20. *l.* 12.

PROP. XII. Problem.

FROM a given Point (A) in any Plain (EF) to erect a Line perpendicular to the ſame Plain. Fig. 19.

From any Point D without the Plain EF make DB (by the foregoing) perpendicular to the Plain EF. And, BA being join'd, draw AC parallel to DB. I ſay the thing is done. The Demonſtration is manifeſt from *Prop.* 8.

Scholium.

IN Practice, from the given Point a Perpendicular is erected to the given Plain, if a Square OKN be applied to the given Point [and be turn'd round.]

PROP. XIII. Theorem.

Fig. 20. *LINES drawn from the ſame Point cannot be both perpendicular to the ſame Plain (AB).*

For, if they were, they would by 6. be parallel. Which cannot be.

PROP. XIV. Theorem.

Fig. 21. *IF the ſame right Line (AB) be perpendicular to two Plains (FG, LQ), the Plains will be parallel.*

Let there be taken in either of the Plains, as FG, any Point C; from which let CE be drawn parallel to AB, and meeting the Plain LQ in E. Then CE (*per* 8. *l.* 11.) will be perpendicular to both Plains FG, LQ. Wherefore, if AC, BE be join'd, the Angles A, B, (by *Def.* 3. *l.* 11.) will be right ones. Therefore (*per* 29. *l.* 1.) AC, BE are parallel. Therefore ACEB is a Parallelogram; and conſequently CE, which hath been already ſhewn to be perpendicular to both Plains, is equal (*per* 34. *l.* 1.) to AB. In the ſame manner I might ſhew, that all the Perpendiculars to both Plains are equal. Therefore (by *Defin.* 8. *l.* 11.) the Plains are parallel. *Q. E. D.*

PROP. XV. Theorem.

Fig 22. *IF two right Lines (BA, CA) touching each other be parallel to two other right Lines which alſo touch one another (ED, FD)*

FD) the Plains likewiſe which are drawn through them will be parallel.

From A let there be drawn AG perpendicular to the Plain EF, and let GH, GI be parallel to DE, DF. Theſe (*per* 9. *l.* 11.) will alſo be parallel to AC, AB. Seeing therefore the Angles IGA, HGA be (by *Defin.* 3. *l.* 11.) right; CAG, BAG will alſo[q] be right Angles. Therefore GA, which is perpendicular to the Plain EF, will alſo be perpendicular to the Plain BC[r]. Therefore the Plains BC, EF are (by the foregoing) parallel. *Q. E. D.*

[q] *Per* 27. *l.* 1.

[r] *Per* 4. *l.* 11.

PROP. XVI. Theorem.

Fig. 23.

A Plain (EHFG) cutting parallel Plains (AB, CD) makes the Sections in them (EH, GF) parallel.

If not, ſeeing they be in the ſame interſecting Plain, they will meet ſomewhere (by *Schol. Prop.* 31. *l.* 1.) as in I. Wherefore ſeeing the whole Lines HEI, FGI be in the Plains[s] AB, CD produc'd, theſe Plains alſo will meet in I. Which is abſurd, and contrary to *Defin.* 8. *l.* 11.

[s] *Per* 1. *l.* 11.

PROP. XVII. Theorem.

Fig. 24.

Parallel Plains cut right Lines (BD and GH) proportionally.

Let the right Lines BH, GD be drawn in the Plains PQ, TV; and likewiſe let BG be drawn meeting the Plain RS in F, and let

FC, FI be join'd. The Plain of the Triangle BGD, cutting parallel Plains, makes the Sections CF, DG parallel (by the foregoing.) Therefore BC is to CD, as BF [t] to FG. Again, the Triangle BHG cutting parallel Plains makes the Sections (by the foregoing) BH, FI parallel. Therefore HI is to IG, as [u] BF to FG; that is, (as I have already ſhew'd) as BC is to CD. *Q. E. D.*

[t] *Per* 2. *l.* 6.

[u] *Per* 2. *l.* 6.

PROP. XVIII. Theorem.

Fig. 25.

IF a right Line (FE) be perpendicular to a Plain (AB), all the Plains which are drawn through it are perpendicular to the ſame Plain (AB.)

Let the Plain GC be drawn through FE, making CD the common Section with AB; and let the Lines HK be drawn in the Plain GC, perpendicular to the common Section CD. Now ſeeing by the Conſtruction HK is perpendicular to the ſame common Section to which FE is perpendicular by the Hypotheſis, KH and FE muſt be parallel (by 29. *l.* 1.) Therefore HK is alſo perpendicular to the Plain AB (*per* 8. *l.* 11.) Therefore the Plain GC is perpendicular to the Plain AB (*per Defin.* 4. *l.* 11.)

PROP. XIX. Theorem.

Fig. 26.

IF two Plains (MF, GD) cutting each other be both perpendicular to the ſame Plain (AB), their common Section alſo will be perpendicular to that Plain (AB).

For ſeeing by the Hypotheſis the Plain MF is perpendicular to the Plain AB; it is manifeſt by Definition 4, that there may be drawn in the Plain MF from the Point L a Perpendicular to the Plain AB. Again, by the Hypotheſis GD is perpendicular to that Plain AB, and therefore there may be alſo drawn in the Plain GD from the Point L a Perpendicular to the Plain AB. But from the Point L [w] there can be erected only one Perpendicular to the ſame Plain AB. Therefore the Perpendicular to the Plain AB, which is drawn from the Point L, muſt be found in both the Plains MF and GD, and conſequently LK, the common Section of thoſe two Plains MF and GD, is perpendicular to the Plain AB. *Q. E. D.*

[w] *Per* 13. *l.* 11.

PROP. XX. Theorem.

IF a ſolid Angle (A) is contain'd under three plain Angles (BAC, CAD, DAB), any two of theſe is greater than the third. Fig. 27.

If the three Angles be equal, the Aſſertion is manifeſt at firſt Sight; and it is as certain, if they be unequal. For let BAD be the greateſt; and from BAD cut off BAE equal to BAC, and make the Line AC equal to AE. And through E let there be drawn a right Line meeting AB and AD in B and D, and let BC, DC be join'd. Becauſe (by the Conſtruction) the Angles BAE, BAC are equal, as likewiſe the Sides BA, AE equal to the Sides BA, AC, the Baſes alſo BE, BC will be equal [x]. And becauſe BC, CD [y] are greater than BD, the Equals BE, BC, being taken away, there

[x] *Per* 4. *l.* 1.

[y] *Per* 20. *l.* 1.

remains CD greater than ED. But the Sides EA, AD are (by the Conſtruction) equal to the Sides CA, AD. Therefore the [*] Angle CAD is greater than the Angle EAD. Seeing therefore the Angle BAC is equal by the Conſtruction to the Angle BAE, thoſe two Angles together BAC, CAD, are greater than the whole BAD. *Q. E. D.*

[*] *Per* 25. *l.* 1.

PROP. XXI. Theorem.

Fig. 28.

THE plain Angles conſtituting any ſolid Angle whatſoever are leſs than four right ones.

Let A be a ſolid Angle; let the right Lines BC, CD, DE, EF, FB, be ſubtended to the plain Angles which make up the ſolid one, ſo that thoſe right Lines be all in one Plain. Which being done, there is conſtituted a Pyramid, whoſe Baſe is the Polygon BCDEF; A is the Top, and it is contain'd under ſo many Triangles G, H, I, K, L, as there are plain Angles which compoſe the ſolid Angle A. And now becauſe the two Angles ABF, ABC, are (by the foregoing) greater than the third FBC; and the two ACB, ACD, are greater than the third BCD, and ſo on: All the Angles of the Triangles G, H, I, K, L, about the Baſe, as taken together, are greater than all the Angles of the Baſe B, C, D, E, F, taken together. But the Angles of the Baſe, together with four right ones, make twice ſo many right Angles (by *Theorem* 2. *Schol.* after 32. *l.* 1.) as there are Sides, or, which is the ſame, as there are Triangles. Therefore all the Angles of the

Triangles about the Baſe, together with four right ones, make more than twice ſo many right Angles, as there are Triangles. But the ſame Angles about the Baſe, together with the Angles that compoſe the Solid, make up [a] twice ſo many right Angles as are the Triangles. It is manifeſt therefore that the Angles which compoſe the ſolid Angle A are leſs than four right ones. *Q. E. D.*

[a] *Per* 32. *l.* 1.

Corollary.

FROM this and the foregoing it is obvious to collect, that a ſolid Angle may be compoſed of any three plain Angles, which are leſs than four right ones, if ſo be that any two of them be greater than the other.

Scholium.

FROM this Propoſition is demonſtrated that famous Theorem, That only three regular and equal plain Figures can contain a Body; to wit, equilateral Triangles, either 4, or 8, or 20. 6 Squares, and 12 Pentagons. And conſequently that there are only five regular Bodies. A Pyramid which is contained under 4; an Octaedrum which is comprehended by 8. And an Icoſiedrum, which is bounded by 20 equilateral Triangles. A Cube which is contain'd under 6 Squares; and the Dodecaedrum under 12 regular and equal Pentagons. Now a Body is called Regular which is comprehended under regular and equal Plains.

Demonſt. A ſolid Angle cannot be compos'd of only two equilateral Triangles; three at leaſt are requir'd.

Of three equilateral Triangles meeting in one Point, there may be conſtituted the ſolid Angle of a Pyramid; of four, the ſolid Angle of an Octaedrum; of five, the ſolid Angle of an Icoſiedrum: Foraſmuch as both 3, 4, and 5 Angles of an equilateral Triangle are leſs than 4 right ones, as is gathered from *Corol.* 12. *prop.* 32. *l.* 1.

And becauſe three Angles of a regular Pentagon (as is gathered from *Coroll. prop.* 11. *l.* 4.) are leſs than four right ones, three Pentagons meeting in one Point will conſtitute a ſolid Angle; that of the Dodecaedrum.

That of three Squares meeting in one Point may be compos'd the ſolid Angle of a Cube, is manifeſt of itſelf. And thus there ariſe five regular Bodies.

But that there are no more than theſe five is thus proved:

Six Angles of an equilateral Triangle make juſt four right ones. For one is two Thirds of one right one; and therefore ſix ſuch will make (by *Coroll.* 12. *prop.* 32. *l.* 1.) twelve Thirds of one right one, that is, four right ones. And therefore of ſix equilateral Triangles a ſolid Angle cannot be compos'd; much leſs of more.

That of four Squares a ſolid Angle cannot be made, much leſs of more, is manifeſt in itſelf.

Four Angles of a regular Pentagon are greater than four right ones. For (by *Coroll. prop.* 11. *l.* 4.) each of them makes ſix Fifths of one right one. Therefore a ſolid Angle cannot be made of four ſuch Pentagons; much leſs of more.

Nor can a ſolid Angle be compos'd of any other regular Figures whatſoever. Three Angles of a regular Hexagon (by *Coroll.* 2. *prop.* 15. *l.* 4.) are equal to four right ones.

For one makes four Thirds of one right one; and therefore three make twelve Thirds of one, that is, four entire right ones. Therefore of three Hexagons a ſolid Angle cannot be made up; much leſs of more.

But, ſeeing three Angles of a regular Hexagon are equal to four right ones, three Angles of any other Figures whatever greater than an Hexagon, as of an Heptagon, Octagon, *&c.* will be greater than four right ones. Wherefore it is manifeſt that the reſt of the regular Figures are all incapable of compoſing a ſolid Angle; and conſequently that there can be no regular Bodies beſides the five foregoing.

PROP. XXII, XXIII.

ARE very prolix, and tedious to Beginners, and ſcarce at any time come into Uſe.

PROP. XXIV. Theorem.

THE Plains which contain a Parallelopiped are (1.) *Parallelograms.* (2.) *The oppoſite ones are equal.* (3.) *The Plains are equal.* Fig. 22.

Part I. The Plain A F cutting the Plains B D, F H, which by *Defin.* 13. are parallel, makes [b] the Sections B A, F E, parallel. Again, the Plain A F cutting the Plains A H, B G, which by the ſame Definition are parallel, (by the ſame) makes the Sections A E, B F, parallel. Therefore B A E F is a Parallelogram. By

[b] *Per* 16. *l.* 11.

By the like Argument the reſt of the Plains of the Parallelopiped may be prov'd to be Parallelograms.

Part II. Becauſe it is manifeſt from the firſt Part, that AB, BC, are parallel to EF, FG, the Angles ABC, EFG, muſt be[c] equal. Wherefore, ſeeing the alternate Sides alſo are equal, the oppoſite Parallelograms BD, FH, are like or ſimilar. And the ſame of the reſt.

[c] *Per* 10. *l.* 11.

Part III. This is manifeſt from the firſt Part, and 4, or 8th of the Firſt Book.

PROP. XXV. Theorem.

Fig. 30. *IF a Parallelopiped* (*GFDI*) *or any Priſm whatever be cut by a Plain* (*NP*) *that is parallel to the oppoſite Sides; there will be this Proportion, as the Baſe* (*DCPO*) *is to the Baſe* (*OPFE*) *ſo is the Solid* (*GP*) *to the Solid* (*NF*).

This is demonſtrated in the ſame manner as *p.* 1. *l.* 6.

Corollary.

A Priſm, cut by a plain Parallel to the oppoſite Plains, hath a Section like and equal to the oppoſite Plains.

PROP. XXVI, XXVII.

A*RE not neceſſary.*

PROP.

PROP. XXVIII. Theorem.

A Plain paſſing through the Diameters of oppoſite Plains (A C, E G) cuts the Parallelopiped into two equal Priſms. Fig. 29.

Becauſe [d] B G, B E, are Parallelograms; C G, A E, are equi-diſtant from the ſame B F. Therefore [e] they are alſo parallel betwixt themſelves, and conſequently are in one Plain. Therefore the right Lines A C, E G, are [f] in one Plain. But now that a Plain drawn thro' them doth cut the Parallelopiped into two equal Priſms is thus ſhew'd: Let the Priſm A E G C D H be underſtood to be ſo conſtituted upon its Plain E A C G, that the Angles D, H, bend towards the Angles B, F. It is manifeſt, that it will yet be betwixt the parallel Plains B A D C, F E H G. But then D muſt needs fall upon B, and H upon F. For, let D fall without B, if it may be, and in N. The Angle B A C [g] is equal to the Angle D C A. But D C A is equal to N A C (for it is one and the ſame Angle). Therefore B A C and N A C are equal: Which is abſurd. Therefore D falls upon B; and for the ſame Cauſe H upon F. Therefore the Priſm A E G C D H coincides with the Priſm A C G E F B, and conſequently they are equal (by *Axiom* 7.)

[d] *Per* 24. *l.* 11.

[e] *Per* 9. *l.* 11.

[f] *Per* 7. *l.* 11.

[g] *Per* 27. *l.* 1.

PROP. XXIX, XXX. Theorems.

THE Parallelopipeds (F E A G K I M C) and F E B H L O M I) which have the ſame Baſe (E F I M) and the ſame Altitude, Fig. 31.

Altitude, and conſequently exiſt between parallel Plains (EFIM) and (GAOL) are equal.

For they either exiſt betwixt the lateral parallel Plains EAOM and FGLI, or not. Let the firſt be ſuppos'd. From the 24th of this, and the 8th of the firſt Book, it is manifeſt that the Triangles AEB, CMO; likewiſe GFH, KIL, are equilateral and equiangular to each other. Wherefore, as in the foregoing, I might ſhew that the Priſms CMOLIK, AEBHFG, being laid upon each other will coincide, and conſequently are equal. Wherefore the common Solid FEBHKCMI being added, the whole Parallelopipeds FEAGKIMC and FEBHLOMI are equal. *Q. E. D.*

Then let the Parallelopiped FXQEMIPR not exiſt betwixt the ſame lateral parallel Plains with the Parallelopiped FEAGKCMI. Here, becauſe by the Hypotheſis, GK, AC, RP, QX, are in one Plain, which is parallel to the Baſe EFIM; let RP, QX, cut GK in L and H, and AC in O and B; and let EB, MO, FH, IL, be join'd. It is eaſy now to ſhew that the Plains containing the Solid FEBHLOMI are Parallelograms, the oppoſite ones of which are equi-diſtant, and conſequently that that Solid is (by *Defin.* 13. *l.* 11.) a Parallelopiped. But to this by the firſt Part the Parallelopipeds FXQEMIPR, and FEAGKCMI, are each of them equal. Therefore they are alſo equal betwixt themſelves. *Q. E. D.*

Corollary.

THIS Propoſition is like to the 35th of the firſt Book; for it affirms concerning Solids, what that doth touching Plains. Wherefore the Demonſtration of the reſt of the Caſes will be like alſo.

PROP. XXXI. Theorem.

Parallelopipeds upon equal Baſes (*A O and E G*) *and in the ſame Altitude* (*S*) *are equal.* Fig. 33.

Firſt, let the Parallelopipeds have their Sides perpendicular to the Baſes. Unto the Side F G produc'd let there be made a Parallelogram G M K H equal and like to the Parallelogram A O; and, the Parallelogram G M P R being perfected, let the right Lines P M, R G meet K H in Q and L. And now let Parallelopipeds be underſtood to be conſtituted upon G K, G Q, G P, whoſe Sides are perpendicular to the Baſes, and S is their common Altitude. The Solid E G S is to the Solid G P S, as E G (*per* 25. *l.* 11.) is to G P; that is (becauſe E G, A O, are equal by the Hypotheſis) as A O to G P; that is, by the Conſtruction, as G K is to G P; that is, as G Q is to G P (*per* 35. *l.* 1); that is, as the Solid G Q S is to the ſame Solid G P S (*per* 25. *l.* 11.) Becauſe therefore the Solids E G S and G Q S have the ſame Proportion to the Solid G P S, the Solid E G S will be equal to the Solid G Q S; that is, to the Solid G K S (*per* 29. *l.* 11.) that is, (becauſe the Baſes G K, A O, are equal and like by the Conſtruction) to the Solid A O S, as it appears from 29. *l.* 11. and even in itſelf. Which was the thing propos'd. *Note*, That in this Reaſoning the Solids are ſuppos'd to be right or perpendicular ones.

Then let the given Parallelopipeds E G S, A O S have their Sides at the Baſes E G, A O, oblique. Let there now be made upon E G, A O, Parallelopipeds, whoſe Sides are perpendicular to the Baſes in the Height S; theſe will be equal to the oblique ones by the 29th or 30th. Wherefore, ſeeing by the firſt Part right Parallelopipeds are equal betwixt themſelves, the oblique ones will be equal betwixt themſelves likewiſe. *Q. E. D.*

PROP. XXXII. Theorem.

Fig. 34. *ALL Parallelopipeds whatever, of equal Height, are betwixt themſelves as their Baſes.*

Let G O and A be the Baſes. Upon C O make the Parallelogram O E equal to A.

Upon B C, O E, let Parallelopipeds be underſtood to be erected in the Altitude K; theſe therefore will be Parts of one Parallelopiped B E K. Therefore the Parallelopiped O E K is to the Parallelopiped B C K, as the Baſe O E to the Baſe B C (*per* 25. *l.* 11.); that is, by the Conſtruction, as the Baſe A is to the Baſe B C. But, becauſe the Baſes O E and A are equal, the Parallelopipeds O E K and A K are equal (by the foregoing). Therefore alſo the Parallelopiped A K is to the Parallelopiped B C K, as the Baſe A is to the Baſe B C. *Q. E. D.*

Scholium.

THAT which hath here been ſhew'd of Parallelopideds, will be demonſtrated in the Twelfth Book of Pyramids, *Prop.* 6; of all Priſms whatever, in *Coroll.* 1, after *Prop.* 9; of Cones and Cylinders, *Prop.* 11.

PROP. XXXIII. Theorem.

LIKE Parallelopipeds (H A and C M) are in a triplicate Proportion of their homologous Sides (A B, B C). Fig. 35.

Let the Parallelopipeds AH, CM be like. Therefore all their Plains (by *Defin.* 9. *l.* 11.) are like; and conſequently AB (by *Defin.* 1. *l.* 6.) is to BC, as EB to BO; and as FB is to BG, ſo is EB to BO. Moreover, the Angles of the Plains are alſo equal (by the ſame). Therefore let the Solids AH, CM be ſo plac'd, that the equal Angles CBO, ABE may be oppoſite, and the Sides AB, CB may lie ſo as to make one ſtraight Line; and then EB, OB will alſo lie ſo as to make one ſtraight Line. Now let Solids be imagin'd to be conſtituted upon the Plains BQ and EC, in ſuch ſort that the Solids KB, HA may be one Parallelopiped, and KB, PO may make one Parallelopiped, and PO, CM may make one Parallelopiped likewiſe. The Solid HA is to the Solid KB (*per* 25. *l.* 11.) as AE to BR; that is, (*per* 1. *l.* 6.) as AB to BC; that is, (as I ſhew'd above by the Hypotheſis) as EB is to BO; that is, (by the ſame) as EC is to BQ; that is, (*per* 25. *l.* 11.) as the ſame Solid KB is to the Solid P O. Therefore the three Solids H A, K B, P O, continue the ſame Proportion. But now the Solid K B is to the Solid P O (by the ſame) as the Baſe B R is to the Baſe P Q; that is, *(per* 1. *l.* 6.) as E B is to B O; that is, as F B is to B G, as it was ſhew'd above by the Hypotheſis; that is, (by the ſame) as the Plain F C is to the Plain B S; that is, *(per* 25. *l.* 11.) as the ſame Solid P O again is to the Solid C M. Therefore the four Solids, H A, K B, P O, C M, are continually proportional. Therefore (by *Defin.* 10. *l.* 5.) the Proportion of the firſt H A to the fourth C M is triplicate of the Proportion of the firſt H A to the ſecond K B; that is, triplicate to the Proportion *(per* 25. *l.* 11.) of A E to B R; that is, triplicate *(per* 1. *l.* 6.) to the Proportion of the homologous Sides, A B to B C. *Q. E. D.*

[Coroll. (1.) *Hence, if there be four right Lines continually proportional; as is the firſt to the fourth, ſo is a Parallelopiped deſcrib'd upon the firſt, to a Parallelopiped like, and in like manner deſcrib'd upon the ſecond.*

(2.) *Upon this alſo depends that moſt famous Problem concerning doubling the Cube; of which afterwards.* Schol. p. 18. *l.* 12.

(3.) *Hence alſo is to be corrected the Error of thoſe, who ſuppoſe that the Proportion of like Solids is the ſame as is that of their Sides. For the Cube of a Line, which is double to another Line, is not only double to the other, but as eight to one. And the Cube of a Line, which is treble to another Line, is not only treble to the other Cube, but contains it* 27 *times. For* 1 : 2 : 4 : 8 ∺ *and* 1 : 3 : 9 : 27 ∺; *and the ſame thing is to be ſaid of all*

all like Bodies whatſoever, as will appear afterwards.

(4.) *Hence the triplicate Proportion of any Quantities whatſoever is the Proportion of the Cubes of the ſame Quantities. Let there be any two Quantities in the triplicate Proportion of the Quantities A B, B C, they ſhall alſo be as A B Cube is to B C Cube.*]

Scholium.

THAT which hath here been ſhew'd of Parallelopipeds will be demonſtrated Book 12; of Pyramids, *Prop.* 8; of all Priſms whatſoever, *Coroll.* 2. *Prop.* 9; of Cones and Cylinders, *Prop.* 12; of Spheres, *Prop.* 18.

PROP. XXXIV. Theorem.

IF the Parallelopipeds (BM, CK) be equal, their Baſes and Altitudes are reciprocally proportional; (that is, the Baſe AM is to the Baſe FK, as reciprocally the Height FC is to the Height AB). Fig. 36.

And, if they be reciprocally proportional, their Baſes and Altitudes are equal.

Part I. Firſt, let the Sides be perpendicular to the Baſes. If now the Altitudes of the Solids B M, C K be equal, the thing is manifeſt.

If the Altitudes be unequal, from the greater F C cut off F E equal to B A; and through E draw the Plain E L parallel to F K. The Baſe A M is to the Baſe F K, (*per* 25. *l.* 11.) as the Solid B M is to the Solid E K; that is, (becauſe by the Hypotheſis the Solids B M, C K are equal) as the Solid C K is to the Solid E K;

that is, (by the ſame) as C G is to E G; that is, *(per* 1. *l.* 6.) as C F is to E F; that is, by the Conſtruction, as C F to B A. *Q. E. D.*

Then let the Sides be oblique to the Baſes. Let right Parallelopipeds be erected upon the ſame Baſes in the ſame Height. The oblique Parallelopipeds will be equal to theſe. Wherefore, ſeeing theſe by the firſt Part have their Baſes and Altitudes reciprocal, thoſe alſo will be likewiſe. *Q. E. D.*

Then let the Sides be oblique to the Baſes. Let right Parallelopipeds be erected upon the ſame Baſes in the ſame Height. The oblique Parallelopipeds will *(per* 29 and 30. *l.* 11.) be equal to theſe: Wherefore, ſeeing theſe by the firſt Part have their Baſes and Altitudes reciprocal, thoſe alſo ſhall be ſo likewiſe. *Q. E. D.*

Part II. Let the Altitudes be unequal, and the Sides perpendicular to the Baſes; and from the greater C F take E F equal to A B. The Solid B M is to the Solid E K, *(per* 32. *l.* 11.) as A M is to F K; that is, by the Hypotheſis, as C F is to A B; that is, by the Conſtruction, as C F is to E F; that is, as C G is to [a] E G; that is, [b] as the Solid C K is to the ſame Solid E K. Therefore the Solids B M and C K have the ſame Proportion to E K. Therefore they are equal. *Q. E. D.*

[a] *Per* 1. *l.* 6.

[b] *Per* 25. *l.* 11.

Corollaries.

WHAT Affections have been demonſtrated of Parallelopipeds, *Prop.* 29, 30, 31, 32, 33, 34, do alſo agree to triangular Priſms, which are the Halves of Parallelopipeds. As is manifeſted from *Prop.* 28. Therefore,

1. The triangular Priſms, which are of equal Height, are as their Baſes, A, B. Fig. 37.

2. If they be like, their Proportion is triplicate to the Proportion of the Sides, oppoſite to equal Angles.

3. If they be equal, they reciprocate their Baſes and Altitudes; and, if they reciprocate their Baſes and Altitudes, they are equal.

Scholium.

WHAT hath here in *Prop.* 34. been ſhew'd of Parallelopipeds, will be demonſtrated in the 12th Book of Pyramids, *Prop.* 9; of all Priſms whatſoever, *Coroll.* 3. after *Prop.* 9; of Cones and Cylinders, *Prop.* 15.

PROP. XXXV.

IS very long, and ſubſervient to the following Propoſition, which we will demonſtrate without it.

PROP. XXXVI. Theorem.

A Parallelopiped (DH) made of three proportional right Lines (A, B, C) is equal to a Parallelopiped (IN), which is made of the Mean (B), and is equiangular to the former. Fig. 38.

Let the Baſe FD of the Parallelopiped DH have the Side EF equal to A, and the other Side ED equal to C: And the Side EG, which ſtands upon the Baſe, equal to B. Thus the

Parallelopiped DH will be made of the three right Lines, A, B, C. Then let the three Sides, LX, IX, XM, (and conſequently all the reſt) of the Parallelopiped IN be equal to the middle Line B, and the ſolid Angle X equal to the ſolid Angle E; the Parallelopiped IN will be made of the Mean B, and be equiangular to the former. I ſay alſo that it is equal.

For, ſeeing by the Hypotheſis and the Conſtruction, as FE is to LX, ſo reciprocally IX is to DE, the Baſes alſo [c] DF, IL will be equal. Now, becauſe the ſolid Angles at E and X are equal, if they be put within one another, [d] they will coincide; and becauſe of the Equality of the right Lines EG, XM, the Points M and G will coincide. Wherefore both the Solids will have one perpendicular Altitude; to wit, the right Line which is let fall from the Points M, G (now become one) unto the Plain of the Baſe. The Solids therefore DH, IN [e] are equal. *Q. E. D.*

[c] *Per* 14. *l.* 6.

[d] *Per Def.* 11. *l.* 11.

[e] *Per* 32. *l.* 11.

Scholium.

WE will further obſerve what is of great Uſe, that of three Lines drawn into, or multiplied one by another after what manner ſoever, a Solid of the ſame Magnitude is produc'd.

ABC. CAB. BCA.
1. 2. 3.

In the preſent Scheme the two firſt Letters deſign the Baſe, the third the Altitude. Let us compare the firſt with the ſecond.

The Baſe AB is to the Baſe CA, *(per* 1. *l.* 6.) as the Side B is to the Side C; that is, reciprocally as the Height B is to the Height C. Therefore, by *p.* 34,

ABC is equal to CAB.

In the ſame manner it may be ſhew'd, that the firſt is equal to the third, and the third to the ſecond.

PROP. XXXVII. Theorem.

Parallelopipeds, which are like, and deſcrib'd in the like manner by proportional right Lines, will themſelves alſo be proportional; and converſely.

This is manifeſt of itſelf. For the Proportions of Parallelopipeds by the 33d of this Book will be triplicate to thoſe Proportions, which by the Hypotheſis are equal, which the Lines have betwixt themſelves.

The Converſe is manifeſt of itſelf alſo.

The Propoſition is true of all Sorts of like Bodies, which will appear from Book the 12th to have betwixt themſelves a Proportion triplicate to that which the Sides have.

PROP. XXXVIII, XXXIX.

THESE contain nothing remarkable, and are ſcarce of any Uſe.

PROP. XL.

THIS is of ſmall Uſe, and indeed no other than the 28th Propoſition in another View.

Scholium.

FROM what hath hitherto been demonſtrated we have the Dimenſion of triangular Priſms, and of quadrangular Parallelopipeds; to wit, if the Altitude be multiplied into the Baſe. As if the Altitude be of 10 Feet, and the Baſe of 100 ſquare Feet, (now the Baſe is meaſured by *Schol. p.* 36, or 41. *l.* 1.) multiply 10 by 100, there will ariſe 1000 cubic Feet for the Solidity of the given Priſm.

The Demonſtration is eaſy. For, like as a Rectangle ariſeth from the Multiplication of one Side by another, ſo a right Parallelopiped is produc'd from the Height drawn into the Baſe. Therefore every Parallelopiped is alſo produc'd from the Altitude multiply'd into the Baſe, ſeeing by 31. *l.* 11. it is equal to a right Parallelopiped, conſtituted upon the ſame Baſe with the ſame Height.

Then ſeeing the whole Parallelopiped is produc'd from the Height into the whole Baſe, the Half of the Parallelopiped (that is, a triangular Priſm by 28. *l.* 11.) will be produc'd from the Altitude multiplied by Half the Baſe; to wit, the Triangle I L K.

THE

THE ELEMENTS OF EUCLID.

BOOK XII.

With us the EIGHTH.

WHAT in the foregoing Books we have endeavoured to perform; namely, to bring the Elements of the Mathematicks into a more eaſy and brief Method, will be to be endeavour'd in this Twelfth Book eſpecially; the Doctrine whereof is moſt neceſſary, but the Demonſtrations are ſo prolix, that they commonly make Beginners almoſt to deſpair. We have ſo propos'd to ourſelves to remedy this Evil, that in the mean while we will not depart from the Rigor of Geometrical Demonſtration.

Which Thing, whether or no we have attain'd, the Reader will underſtand, if he ſhall compare this of ours with *Euclid*'s Prolixity.

Now, after Euclid *had in the former Book declared the Elements of Solids, and defined the Meaſures of the moſt eaſy Bodies, thoſe, namely, which are terminated with plain Surfaces; in this Twelfth Book he conſiders Bodies bounded with curve Surfaces; to wit, Cylinders, Cones, and Spheres; compares them betwixt themſelves, and defines their Meaſures. This Book is indeed moſt profitable, becauſe it contains thoſe Principles, on which the chief Maſters of Geometry, and eſpecially* Archimedes, *have built ſo many famous Demonſtrations concerning the Cylinder, Cone, and Sphere.*

DEFINITIONS.

Fig. 1. *l.* 12. 1. A Pyramid is a Solid (Z L) comprehended under the Triangles (A L C, C L F, F L B, B L A) plac'd from one Plain (Z) to one Point (L).

The Plain Z is called the Baſe, and may be either a Triangle or Quadrangle, or any other Figure; from each of the Sides whereof there ariſe Triangles meeting together in the Point L, which is call'd the Vertex or Top.

As the Triangle amongſt rectilineal plain Figures, ſo the Pyramid amongſt ſolid ones is the firſt and moſt ſimple.

Fig. 2, 3. 2. If without the Plain of ſome Circle (C L) there ſhall be taken the Point (A), and from it be drawn the infinite right Line (A F) touching the Circle in C; and this Line, the Point (A) remaining fix'd, be turn'd about the Circumference of the Circle, until it returns thither from whence it began to be moved; the Surface deſcrib'd by the right Line (A C F) is term'd a conical Surface, and the Body which is contain'd under this Surface, and the Circle (C L) is call'd a Cone.

The Vertex of the Cone is (A).

The Circle (C L) is the Baſe of the Cone.

The right Line (A B) drawn from the Vertex to the Centre of the Baſe is the Axis of the Cone.

The Side of the Cone is the right Line (A C) drawn from the Vertex to the Circumference of the Baſe, which that it is wholly in the Surface of the Cone is manifeſt from the Production of the Figure.

A right[f] Cone is when the Axis (A B) is perpendicular to the Baſe. [f] Fig. 3.

A Scalene[g] or oblique Cone, is when the Axis (A B) is not perpendicular to the Baſe. [g] Fig. 4.

A right Cone is alſo made by a right-angled Triangle (C B A) turn'd round about one of the perpendicular Sides (A B). See *Fig.* 2.

3. If an infinite right Line (C O F) be turn'd about two Circles (C L, O Q) equal and parallel, until it returns to that Place from whence it began to be mov'd, and remains always, whilſt it is mov'd, parallel to itſelf, the Surface deſcrib'd by the right Line (C O F) is called a cylindrical Surface; and the Body which is contain'd under this Surface, and the two Circles, is called a Cylinder. Fig. 4, 5.

The Baſes of the Cylinder are the Circles (C L, O Q) the right Line (A B) which connects the Centres of the Baſes, is called the Axis. The right Line (O C) in the Surface of the

Cylinder, touching both the Baſes, is called a Side of the Cylinder.

Fig. 4. A right Cylinder is when the Axis is perpendicular to the Baſe.

Fig. 5. A Scalene, or oblique Cylinder, is when the Axis is not perpendicular to the Baſe.

A right Cylinder is alſo made by a Rectangle (O C B A) turn'd round about one Side (B A). See *Fig.* 4.

Fig. 20, 21. 4. Like Cones and Cylinders are thoſe, which have their Axes (A K, Z O) and the Baſes of their Diameters (B F, Q R) proportional.

5. A Sphere is a Solid contain'd under one Surface, unto which Surface all the right Lines, that are drawn from a certain Point within the Figure, are equal amongſt themſelves. That Point is call'd the Centre. The Diameter of the Sphere is a right Line drawn through the Centre unto the Surface on both Sides.

Fig. 6. A Sphere is produced if a Semicircle be turn'd about its Diameter (A F) which remains in the mean while unmov'd.

6. Magnitudes inſcrib'd into or deſcrib'd about ſome Figure, whether they be greater or leſſer than the Figure, are then ſaid to *end* in the Figure, when they will at the laſt differ from it by a Quantity leſs than any given one whatſoever, or how ſmall ſoever.

Therefore if thoſe Magnitudes which are inſcrib'd into ſome Figure will at laſt fall ſhort of it by a Deficiency leſs than any given one whatſoever, the Magnitudes inſcrib'd are ſaid to *end* in the Figure; and if thoſe which are circumſcrib'd about ſome Figure will at laſt exceed it by an Exceſs leſs than any given one whatſoever, they ſhall be ſaid to *end* in the Figure.

PROPOSITION I. Theorem.

THE *Proportion of like Polygons inſcrib'd in a Circle is duplicate to the Proportion of the Diameters (AF, IC).* Fig. 6, 7.

Let AO, BF; IR, LC, be drawn. Becauſe the Polygons are ſuppos'd to be like, the Angles (OBA, RLI) will (*per Defin.* 1. *l.* 6.) be equal; and the Sides OB, BA, proportional to the Sides RL, LI. Therefore in the Triangles OAB, RIL (*per* 6. *l.* 6.) the Angles O and R are equal. Therefore alſo BFA and LCI, which ſtand upon the ſame Arches BA, LI, are (*p.* 21. *l.* 3) equal. But the Angles FBA, CLI, in Semicircles, are (*p.* 31. *l.* 3.) right ones. Therefore the other Angles (*per Coroll.* 9. *pr.* 32. *l.* 1.) BAF, LIC, are equal. Therefore becauſe the Triangles FAB, CIL, are equiangular to each other, they are (*p.* 4. *l.* 6.) like; and BA will be to LI, as AF to IC. Now, becauſe by the Hypotheſis the Polygons are like, their Proportion will be duplicate (*p.* 20. *l.* 6.) to the Proportion of the Sides BA, LI; that is, as I have already ſhew'd, duplicate to the Proportion of the Diameters AF, IC. *Q. E. D.*

Corollary.

THE Circumferences of like Polygons inſcribed in a Circle are betwixt themſelves as the Diameters. Fig. 6, 7.

Seeing it hath already been ſhew'd, that AB is to LI, as AF is to IC; OB will alſo be to R L, as A F to I C: and ſo of the reſt of the Sides. Therefore all the Sides together will be to all the Sides together, that is, one Circumference to another, as A F is to I C.

A Lemma.

Fig. 8. POLYGONS inſcrib'd in a Circle end in a Circle. Inſcribe a Square, as A C BD. Seeing this is half (*per Schol. p.* 6, and 7. *l.* 4.) of the Square which is circumſcrib'd, it will be greater than half of the Circle. Wherefore if this be taken out of the Circle, there will be taken out of it more than half. Then each Arch being biſected in E, K, I, H, inſcribe an Octagon: and let F G touch the Circle in E, which F G let B C, D A meet in G and F; C F will be a Parallelogram, of which ſeeing the Triangle C E A (*per* 41. *l.* 1.) is half, this will be more than half of the Segment C E A. In the ſame manner each of the Triangles A K D, D I B, *&c.* is more than half each of the Segments. Therefore all the Triangles are more than half all the Segments. Therefore if you take theſe out of thoſe, that is, out of the Remainder of the Circle, more than half will be taken away. In the ſame Way of arguing, if there be inſcrib'd in the Circle Polygons of Sides always double in Number; I can ſhew that there will always be taken out of the Remainder of the Circle more than half. Therefore the Remainder muſt at laſt be leſs than any given one whatſoever; and conſequently the inſcrib'd Polygons will at laſt fall ſhort of a Circle by a Quantity leſs than any given one whatſoever; that is (*per Defin.* 6. *l.* 12.) will end in a Circle.

PROP.

PROP. II. Theorem.

THE Proportion of Circles is duplicate to the Proportion of their Diameters. Fig. 6, 7.

The Proportion of Polygons inſcrib'd in a Circle without End is (*per* 1. *l.* 12.) duplicate to the Proportion of the Diameters. But Polygons (by the foregoing *Lemma*) inſcrib'd in a Circle infinitely, at laſt end in the Circle. Therefore the Proportion of Circles is alſo duplicate to the Proportion of the Diameters.

PROP. III, IV.

ARE prolix, and hard for young Beginners, and have no other Uſe, than that they ſerve to the Demonſtration of the Fifth, which we ſhall demonſtrate much more eaſily without them.

Lemmata, *or preparatory Propoſitions to* Prop. V.

Lemma I.

IF two triangular Pyramids be cut with Plains (OSE, RXZ) parallel to the Baſes (ABC, IQV), which ſame Plains divide the Sides (CF, QL) proportionally in (E and Z) (OSE, RXZ) will be betwixt themſelves as the Baſes (ACB, IQV). Fig. 9.

Becauſe the parallel Plains OSE, ABC, are cut by the Plains BFC, AFB, AFC, the common Sections SE, BC, and OS, AB, and

OE, AC will be *(per* 16. *l.* 11.) parallel. Wherefore the Angles OSE, ABC, and SOE, BAC, and OES, ACB, two and two, are *(per* 10. *l.* 11.) equal. Wherefore the Sections OSE, ABC are like *(per* 4. *l.* 6.) In the ſame manner I might ſhew, that the Sections RXZ, IVQ are like. Therefore *(per* 19. *l.* 6.) the Proportion of the Section ABC, to the Section OSE, is duplicate to the Proportion of the Side BC, to the Side SE; and the Proportion of the Section IVQ to RXZ is duplicate to the Proportion of VQ to XZ. But the Proportions of BC to SE, and of VQ to XZ, are the ſame, (for BC is to SE (by *Coroll.* 1. *per* 4. *l.* 6.) as CF to EF; that is, by the Hypotheſis, as QL to ZL; that is, (by the ſame *Coroll.)* as VQ to XZ). Therefore the Proportion of ABC to OSE is the ſame with the Proportion of IVQ to RXZ. *Q. E. P.*

Lemma II.

Fig. 10. PRiſms inſcrib'd infinitely in a Pyramid (ZCAF) which hath a triangular Baſe, end in the ſame Pyramid.

Let the Side of the Pyramid be divided into a certain Number of equal Parts, AB, BG, GF; and, through B and G there being made the Sections GDN and BEP parallel to the Baſe ZAC, let the triangular Priſms BEPMAO and GDNKBQ be underſtood to be inſcrib'd in the Pyramid. Theſe then being continued without the Pyramid, let there be underſtood to be deſcrib'd about the Pyramid the Priſms CIBA, PXGB, NHFG. The Exceſſes of the circumſcrib'd Priſms above the inſcribed ones are the Solids I M, X K, H G, which taken together are equal to the Priſm C I B A: For H G (*per* 25. *l.* 11.) is equal to D B; and conſequently H G with X K are equal to P X G B, that is (by the ſame) to M E B A. Therefore the three H G, X K, I M, are equal to the whole C I B A. But if A F be divided without End into more equal Parts, and conſequently the Number of Priſms be infinitely increas'd, A B will become leſs than any given Line. Therefore (as is manifeſt from *p.* 25. *l.* 11.) the Priſm C I B A will become leſs than any given one. Therefore the Exceſs of the circumſcrib'd Priſms (and much more of the Pyramid Z C A F, which is Part of the Priſms circumſcrib'd about it) above the inſcribed Priſms will be leſs than any given Priſm. Therefore the inſcrib'd Priſms (by *Defin.* 6. *l.* 12.) end at laſt in a Pyramid. *Q. E. D.*

PROP. V. Theorem.

TRiangular Pyramids of the ſame Height have that Proportion betwixt themſelves, which their Baſes (AQR, ESX) have. Fig. 11.

Let the equal Altitudes of the Pyramids be repreſented by the Sides A P, E Z; which on both Sides let be divided into as many equal Parts as you will, but ſo that they be of the ſame Number; and let there be made through the Points of the Diviſions Sections parallel to the Baſes; let triangular Priſms, of the ſame Number and the ſame Height, be underſtood to be inſcrib'd in both Pyramids. And now becauſe the Priſms L A, I E are of the ſame

Height, the Priſm LA will be to the Priſm IE (by *Coroll.* 1. *p.* 34. *l.* 11.) as the Baſe LOB is to the Baſe INK; that is, (by *Lemma* 1.) as the Baſe QRA is to the Baſe SXE. In the ſame manner I might ſhew, that each of the Priſms, inſcrib'd into the Pyramid QPAR, is to each inſcrib'd into the Pyramid SZEX, as the Baſe QAR is to the Baſe SEX. Therefore all of them together are to all of them together, as Baſe is to Baſe. Wherefore, ſeeing they at laſt end (*per Lem.* 2.) in the Pyramids themſelves, the Pyramids themſelves alſo will be as their Baſes. *Q. E. D.*

PROP. VI. Theorem.

Fig. 12, 13.

ALL Pyramids whatſoever, which are of equal Height, have that Proportion betwixt themſelves which their Baſes (AB, CFO) have.

Let their Baſes be reſolv'd into Triangles A, B, C, F, O; and the whole Pyramids into triangular Pyramids. The Pyramid AX is to the Pyramid OZ (by the foregoing) as A is to O; and the Pyramid BX is to the Pyramid OZ, as B is to O (by the ſame.) Therefore the Pyramids AX, BX together (that is, the whole Pyramid ABX) are to the Pyramid OZ, as A, B together are to O. By the ſame Argumentation the Pyramid ABX is to the Pyramid FZ (by the foregoing) as A, B are to F: And ABX is to CZ, as A, B is to C. Therefore ABX is to the three OZ, FZ, CZ together; that is, to the whole Pyramid OFCZ, as A, B together is to O, F, C together. *Q. E. D.*

PROP. VII. Theorem.

EVERY Pyramid is the third Part of a Priſm which hath the ſame Baſe and Height.

Firſt, Let the triangular Pyramid BGAC have the ſame Baſe and Height with the Priſm BACFEO: Let BF, AO, AF be drawn. The Triangles BFC, BFO are *(per* 34. *l.* 1.) equal. Therefore the Pyramid BFCA is equal to the Pyramid BOFA. For the ſame Reaſon OEAF is equal to the Pyramid OBAF; that is, to the Pyramid BOFA, for they are the ſame Pyramids. Therefore BFCA and OEAF are alſo equal. Therefore all three, BFCA, OEAF, OBAF, or BOFA, are equal. Therefore the three together are triple of one BFCA. But thoſe three conſtitute the Priſm BACFEO. That Priſm therefore is treble to the Pyramid BFCA; that is *(per* 5. *l.* 11.) to BGAC. *Q. E. D.* Fig. 14.

Then let any Pyramid whatſoever have the ſame Baſe and Height with the Priſm AEFH: The Lines BC, BO, BE, and NI, NG, NH being drawn, reſolve the Priſm into triangular Priſms, and the Pyramid into triangular Pyramids. Which being done, the Demonſtration is manifeſt from the firſt Part. For each Part of the Priſms will be treble of each Part of the Pyramids. And conſequently the whole Priſm will be treble to the whole Pyramid. *Q. E. D.* Fig. 15.

PROP. VIII. Theorem.

Fig. 16. *THE Proportion of like Pyramids (OACB, KHIN) is triplicate to that which the homologous Sides (AB, HN) have to each other.*

Firſt, Let them be triangular: The Parallelograms AM and HQ being perfected, ſet upon them the Parallelopipeds AG, HL, in the Height of the Pyramids; which, ſeeing the Pyramids are like, will alſo (as appears from *Defin.* 9. *l.* 11.) be like. Then let EF, RP be drawn; and through EF, CB, as likewiſe through RP, IN, the Parallelopipeds will be cut (*per* 28. *l.* 11.) into two equal Priſms; each of which will be treble to the Pyramids OACB and KHIN (by the foregoing.) Therefore both together, that is, the whole Parallelopipeds AG, HL will be ſixfold of the Pyramids. Therefore the Pyramids are proportional to the Parallelopipeds. But (*per* 33. *l.* 11.) the Proportion of theſe each to other is triplicate to the Proportion of the Sides AB, HN. Therefore ſo likewiſe is the Proportion of the Pyramids.

Fig. 17. But, if the like Pyramids ſhall be polygonal, let them be reſolv'd into the triangular ones AR, BR, CR, and OK, EK, FK. You may from 20, and 5. *l.* 6. and *Defin.* 9. *l.* 11, eaſily ſhew, that AR is like to OK, and BR to EK, and CR to FK. Therefore, by the former Part, the Proportion of the Pyramids AR, OK, is triplicate to the Proportion of IM to PZ: And the Proportion of the Pyramids BR and EK is triplicate to the Proportion of MX to SZ; that is, again by the Hypotheſis of IM to PZ; and the Proportion of the Pyramids CR, FK is triplicate to the Proportion of XQ to ST; that is, again of IM to PZ. Seeing therefore the Proportion of each to each is triplicate to the Proportion of IM to PZ, the Proportion alſo of all to all (that is, the Proportion of the whole Pyramid ABCR to the whole OEFK) will be triplicate to the Proportion of IM to PZ. *Q. E. D.*

PROP. IX. Theorem.

Equal Pyramids have their Baſes and Altitudes reciprocally proportionable; and thoſe which have them ſo are equal. Fig. 18, 19.

Part I. Firſt, Let the Pyramids be triangular BACO, KHNL: The Parallelograms BE, HR being perfected, upon theſe ſet the Parallelopipeds BF, HP. Theſe will be (as was ſhew'd in the foregoing) ſixfold of Pyramids, which are by the Hypotheſis equal, and conſequently will be equal betwixt themſelves. But now the Altitudes of theſe Parallelopipeds HK, BA, are the ſame with thoſe of the Pyramids; and the Baſes BE, HR, are double to *(per 34. l. 11.)* the pyramidal Baſes BCO, HNL, and conſequently proportional to them. Seeing therefore by reaſon of the Equality of the Parallelopipeds, as BE is to HR, ſo (by the ſame) is reciprocally HK, to BA; it will alſo be that, as the Baſe BCO is to the Baſe HNL, ſo reciprocally is the Altitude HK to the Altitude BA. *Q. E. D.*

But, if the Pyramids have polygonal Baſes, let them be reduced into triangular ones, retaining the ſame Altitudes; and theſe will be equal to thoſe by the 6th. But the Pyramids thus reduc'd have, as we have now demonſtrated, their Baſes and Altitudes reciprocally proportionable. Therefore the given polygonal Pyramids alſo have their Baſes and Altitudes reciprocally proportional. *Q. E. D.*

Part II. Becauſe it is now ſuppos'd, that BCO is to HLN, as HK is to BA; BE will alſo be to HR, as HK is to BA. Therefore the Parallelopipeds BF, HP are (*per* 34. *l.* 11.) alſo equal. Therefore their ſixth Parts alſo, to wit, the Pyramids BACO, HKNL are equal. *Q. E. D.*

Corollaries.

WHAT has been demonſtrated of Pyramids in *Pr.* 6, 8, 9, does alſo agree to all Priſms whatſoever; ſeeing theſe are (*per* 7. *l.* 12.) treble to Pyramids which have the ſame Baſes and Altitudes. Therefore,

1. In Priſms of the ſame Height their Proportion is the ſame as that of their Baſes. For this was ſhew'd of Pyramids, *Pr.* 6.

2. The Proportion of like Priſms is triplicate to the Proportion of their homologous Sides. For this was ſhew'd concerning Pyramids, *Pr.* 8.

3. Equal Priſms have their Baſes and Altitudes reciprocally proportionable; and thoſe which have them ſo are equal. For this is ſhew'd of Pyramids, *Pr.* 9.

It is ſtrange that theſe things were paſs'd over by *Euclid*, ſeeing they are the chief things which can be delivered concerning rectilineal Solids.

Scholium.

FROM what has been hitherto demonſtrated is drawn the Method of meaſuring any Priſms or Pyramids whatſoever.

The Solidity of a Priſm is produc'd from the Altitude multiplied into the Baſe, and that of a Pyramid from the third Part of the Altitude multiplied by the Baſe.

As if the Altitude of a Priſm be of 5 Feet, but the Baſe contains 25 ſquare Feet; multiply 25 by 5, and there ariſe 125 cubic Feet for the Solidity of the Priſm.

For let there be a polygonal Priſm, as A H. And let the Triangle B A C be underſtood to be equal to its Baſe A E, and upon B A C the Priſm B E to be ſet at equal Height with A H. The Priſms B E, A H will be (by *Coroll.* 1. foregoing) equal. But the Priſm B E (by *Schol. p.* 40. *l.* 11.) is produc'd from its Altitude drawn into the Baſe B A C; that is, into A E, by Conſtruction. Therefore the Priſm A H alſo is made of its Baſe A E multiplied by its Height, which is ſuppos'd to be equal to the Height of the Priſm B E. Fig. 14, 15.

From hence, and from 7, the Demonſtration of the ſecond Part is alſo manifeſt.

A Lemma to Prop. 10.

PYramids and Priſms, which are inſcrib'd in Cones and Cylinders infinitely, do at laſt end in the Cones and Cylinders.

This is demonſtrated as the *Lemma* of *Prop.* 2. with the Help of *Prop.* 6. and of *Coroll.* 2. after *Prop.* 9. if, as there, Plains inſcrib'd in a Circle, ſo here Priſms and Pyramids which ſtand upon thoſe Plains as their Baſes, be continually taken away from the Cones and Cylinders.

PROP. X. Theorem.

Fig. 20. *EVery Cone is a third Part of a Cylinder, having the ſame Baſe and Height.*

Let a regular Polygon, of as many Sides as you pleaſe, be underſtood to be inſcrib'd in the Baſe CL; and upon it, as the Baſe, for a Cone let a Pyramid, and for a Cylinder a Priſm be inſcrib'd. The Pyramid *(per* 7. *l.* 12.) will be a third Part of the Priſm. And if again in the Circle a Polygon of twice as many Sides be inſcrib'd, and upon it be inſcrib'd for a Cone a Pyramid, but for the Cylinder a Priſm, the Pyramid will again be a third Part of the Priſm. And thus it will always be. Wherefore ſeeing Pyramids end in a Cone, and Priſms in a Cylinder, the Cone alſo will be a third Part of the Cylinder. *Q. E. D.*

PROP. XI. Theorem.

Fig. 20, 21. *CONES of equal Height (BAF, QXR) are as their Baſes (CL, SE). The ſame thing belongs to Cylinders of equal Height alſo.*

Pyramids, inſcrib'd into Cones of equal Height, are as their *(per* 6. *l.* 12.) Baſes. But Pyramids do at length end in Cones. Therefore

Cones alſo are as their Baſes. And ſeeing Cylinders are threefold of Cones, which have the ſame Baſe and Altitude with them, they alſo will be as their Baſes. *Q. E. D.*

Corollary.

IN the ſame manner it may be demonſtrated that alſo Priſms and Cylinders of equal Height are betwixt themſelves as their Baſes; yea, that all cylindrical Bodies of the ſame Altitude, that is, which are produc'd from whatſoever Plains multiplied by the ſame Altitude, are betwixt themſelves as their Baſes. You may reaſon in the ſame manner of Pyramids and Cones of equal Altitude, and of all conical Bodies whatſoever.

PROP. XII. Theorem.

THE Proportion of like Cones (BAF and QZR) is triplicate to the Proportion of the Diameters (BF and QR) which are in the Baſes. The ſame Thing is to be ſaid of like Cylinders. Fig. 20, 21.

In the Baſes of the like Cones let regular Polygons be inſcribed, which Polygons conſequently will be like. The Pyramids which are inſcribed upon theſe Polygons will alſo be like; as may be eaſily ſhew'd. Therefore their Proportion is triplicate (*per* 8. *l.* 12.) to the Proportion of the Sides B L, Q E; that is, to the Proportion of the Diameters B F, QR. Wherefore, ſeeing the Pyramids end in Cones, the Proportion alſo of the Cones is triplicate to the Proportion of the Diameters B F, QR. *Q. E. D.*

The Theorem is manifeſt of Cylinders, ſeeing they are treble to Cones.

PROP. XIII. Theorem.

Fig. 22. *IF a Cylinder (B I) be cut with a Plain (R L) parallel to the Baſes (B Q, C I); one Part (B L) ſhall be to the other Part (R I), as one Segment of the Axis (A O) is to the other Segment of the Axis (O F).*

This Propoſition is demonſtrated as the firſt of *l.* 6.

The Theorem is in the ſame manner true of the Superficies.

PROP. XIV. Theorem.

Fig. 23, 24. *CYLINDERS (AR and CI) of equal Baſes (M Q, G H) are as their Altitudes (L Z, S F). The ſame Thing happens to Cones.*

Cut off from the higher Cylinder A R the Cylinder A O, whoſe Height L E is the ſame with S F. Therefore (*per* 11. *l.* 12.) the Cylinders A O, C I, are equal. Seeing therefore the Cylinder A O is to the Cylinder A R (by the foregoing) as L E is to L Z; C I alſo ſhall be to A R as L E is to L Z; that is, (becauſe L E and S F are equal, by Conſtruction) as S F to L Z. *Q. E. D.*

Corollary.

THE Theorem is alſo true of Priſms, and likewiſe of Pyramids, and the Demonſtration altogether alike. But of Priſms the Thing is demonſtrated from *Corol.* 1. *p.* 9. *l.* 12, and 25. *l.* 11. and its *Corol.* Of Pyramids from this, and from *p.* 7. *l.* 12.

PROP. XV. Theorem.

EQUAL Cylinders (AR, DF) have their Baſes and Altitudes reciprocally proportional; and, if they have them ſo, they are equal. The ſame Thing is true of Cones. Fig. 24, 25.

This is demonſtrated as *Prop.* 34. *l.* 11. only for 32, and 25. *l.* 11. there cited, there muſt be cited here *Prop.* 11, and 13. *l.* 12.

Scholium.

WHereas *Euclid* hath ſaid nothing of compound Proportion in Bodies, we ſhall briefly demonſtrate it in this Place.

1. A Cylinder hath to a Cylinder, and a Priſm to a Priſm, a Proportion compounded of the Proportions of the Baſes and Altitudes.

Let FD and AR be Cylinders of different Altitudes (for in thoſe of equal Altitude the Thing is manifeſt). From the higher cut off AO of equal Height with FD. And let the Proportion be thus; as the Baſe UT is to the Baſe MQ, ſo FN to X; and as the Altitude ND or BO is to the Altitude BR, ſo is X to Z. We muſt therefore ſhew, that the Cylinder Fig. 24, 25.

der F D is to the Cylinder A R, as F N is to Z. The Cylinder F D is to the Cylinder A O (*per* 11. *l.* 12.) as the Baſe V T is to the Baſe M Q; that is, (by Conſtruction) as F N is to X; but the Cylinder A O is to the Cylinder A R (*per* 13. *l.* 12.) as B O to B R; that is (by Conſtruction) as X to Z. Therefore by Proportion of Equality the Cylinder F D is to the Cylinder A R, as F N to Z.

The Propoſition may be demonſtrated in the ſame manner of Priſms, but from *Coroll.* 1. *pr.* 9. and *Coroll. pr.* 14.

2. A Cone alſo hath to a Cone, and a Pyramid to a Pyramid, a Proportion which is compounded of the Proportions of Baſe to Baſe, and Altitude to Altitude.

For (by *Prop.* 10. and 7. *l.* 12.) they are third Parts of Cylinders and Priſms.

PROP. XVI, XVII.

THESE Propoſitions, the moſt prolix of all others, have no other Uſe than to ſerve to the demonſtrating Prop. 18. *which we ſhall demonſtrate in another more eaſy Way.*

Lemma to Prop. 18.

Fig. 26. CYLINDERS inſcrib'd in an Hemiſphere end in the Hemiſphere. Let P Z Y be the greateſt Semicircle of the Hemiſphere; and let the Radius A Z be perpendicular to the Diameter P Y. Cut A Z into a certain Number of equal Parts, A M, M N, N Z; and there being drawn through the Points of the Diviſions M, N, the perpendicular Lines

B O, *&c.* Let there be inſcrib'd in the Semicircle the Rectangles OBRK, EDHS; which afterwards being continued without the Semicircle, let there be underſtood to be deſcrib'd about the Semicrcle the Rectangles F T Y P, LVBO, QXDE. They will all of them be of the ſame Height, and the Exceſſes of the circumſcribed ones above thoſe which are inſcrib'd will be the Plains FK, LS, XE, VH, T R, which taken together make the Rectangle F T Y P. For, becauſe X E is equal to D S, thoſe LS, VH, XE, together will be equal to the Rectangle LB, that is, O R. Wherefore, if you add on both Sides the Plains FK, T R, all thoſe FK, LS, XE, VH, TR, taken together, will be equal to the Rectangle F T Y P. If now the Semicircle with the Rectangles be underſtood to be turn'd about the Radius A Z, which is in the mean while unmov'd, the inſcribed Rectangles E H, O R, will produce Cylinders inſcrib'd in the Hemiſphere; and the circumſcrib'd Rectangles will produce Cylinders circumſcrib'd about the Hemiſphere, ſtanding one upon another; and as the Exceſſes of the circumſcribed Rectangles above the inſcribed ones was the Rectangle F Y; ſo likewiſe the Exceſſes of the circumſcribed Cylinders, above the inſcribed ones, will be the Cylinder which is produced from the Rectangle F Y. But now the Altitude of this Cylinder will be made leſs than any given Height; and conſequently (as is manifeſt from 13. *l.* 12) itſelf will grow to be leſs than any given Cylinder, if, the Radius being divided into more equal Parts without End, the Number of Rectangles, and from thence of Cylinders, be infinitely increas'd. Therefore the Exceſs of the circumſcrib'd Cylinders, and much more of the Hemiſphere itſelf, which is only a Part of the circumſcrib'd Cylinders, above the inſcribed ones, will at laſt become leſs than any given one. Thereſore (by *Defin* 6. *l.* 12.) Cylinders infinitely inſcribed in an Hemiſphere do at length end in the Hemiſphere itſelf. *Q. E. D.*

Corollary.

IN the ſame manner it will be demonſtrated, that Cylinders inſcrib'd in a Cone, Conoid, Spheroid, *&c.* do at laſt end in the ſame.

PROP. XVIII. Theorem.

Fig. 27. T*HE Proportion of Spheres is triplicate to the Proportion of their Diameters (BK, RZ).*

The Radius's AB, YR, being divided into as many equal Parts as you will, but of an equal Number, and there being drawn through the Points of the Diviſions perpendicular, *&c.* Let Rectangles of an equal Number be underſtood to be inſcribed in the greateſt Semicircles of the Spheres, which Rectangles being turned about the unmov'd Radius's AB, YR will be conceiv'd to inſcribe in both the Hemiſpheres a like Number of Cylinders ſtanding one upon another. Now becauſe KC is (*per Coroll. p.* 13. *l.* 6.) to CF, as CF is to CB; the Proportion of KC to CB (by *Defin.* 10. *l.* 5.) will be duplicate to that of KC to CF, that is, to the Proportion of FC to CB. In like manner the Proportion of ZE to ER will be duplicate to the Proportion of XE to ER. But by the Conſtruction KC is to CB as ZE is to ER. Thereſore FC alſo is to BC as

XE to ER. But BC by the Conſtruction is to CO as RE to ES. Therefore by Equality FC is to CO, as XE is to ES. Therefore (by *Defin.* 4. *l.* 12.) the Cylinders FL, XQ, are like, and conſequently their Proportion is triplicate to (*per* 12. *l.* 12.) the Proportion of their Diameters, FI, XV, or of the Semidiameters FC, XE, which are the Baſes. But the Proportion of FC to XE is the ſame with the Proportion which is betwixt the Diameters of the Spheres BK, RZ; (for as I have already ſhew'd FC is to XE as CO is to ES; that is, as BK is to RZ which by the Conſtruction are Equi-multiples of thoſe CO, ES). Therefore the Proportion of the Cylinders FL, XQ is triplicate to the Proportion of the Diameters BK, RZ. In the ſame manner we might demonſtrate that each Cylinder, inſcribed in one Hemiſphere, bears to each Cylinder inſcribed in the other Hemiſphere a Proportion triplicate to the Proportion of the Diameters BK, RZ. Therefore alſo the Proportion of all together to all together is triplicate to the Proportion of the Diameters BK, RZ. Wherefore, ſeeing the Aggregates of the Cylinders do at length end in their Hemiſpheres, the Proportion of the Hemiſpheres alſo, and conſequently of the Spheres, will be triplicate to the Proportion of their Diameters. *Q. E. D.*

Corollary.

THerefore the Proportion of the Diameters being known, the Proportion of the Spheres becomes known likewiſe. As if the Diameter of the leſſer be one Foot, that of the greater ten Feet; let the Proportion of one to ten be continued through four Terms, 1, 10, 100, 100, 1000. as 1 the firſt is to 1000, the 4th Term, ſo is the leſſer Sphere to the greater.

The Dimenſion of Cones, Cylinders, and of the Sphere, will be exhibited in the following Book out of *Archimedes*.

Scholium.

AS like plain Figures are increas'd or diminiſh'd in any given Proportion by one mean Proportional, ſo like Bodies are increas'd or diminiſh'd by two mean Proportionals.

Let a Sphere or Cube, or any other Body whatſoever, be given, whoſe Radius or Side is A. Likewiſe let any Proportion whatſoever of A to B be given, as the Double or 2 to 1. A Body is to be diſcover'd both double to the given one, and like to it.

Betwixt the Terms of the given Proportion A and B, let there be found two mean Proportionals X, Z, according to what was taught in the *Scholium* of *Prop.* 13. *l.* 6. A Sphere whoſe Radius is X, or other Body like to the given one which is made upon the Side X, will be double to the given one.

For like Bodies, whoſe Radius's or Sides are A and X have betwixt themſelves a Proportion which is triplicate to the Proportion of A to X (by *Coroll. Prop.* 9, and by *Prop.* 12 and 18. *l.* 12.) that is, the ſame *(per Defin.* 10. *l.* 5.) which A hath to B.

And this is that moſt celebrated Problem, which from *Apollo* of *Delos* is called the *Deliacal* Problem; becauſe at the time of a moſt grievous Peſtilence, which waſted *Athens*, being conſulted, he gave Anſwer, That the Peſtilence would ceaſe, if his Altar, which was of a cubical Form, were doubled. Thus *Valerius Maximus*, l. 8.

THEO-

THEOREMS

Selected out of

ARCHIMEDES:

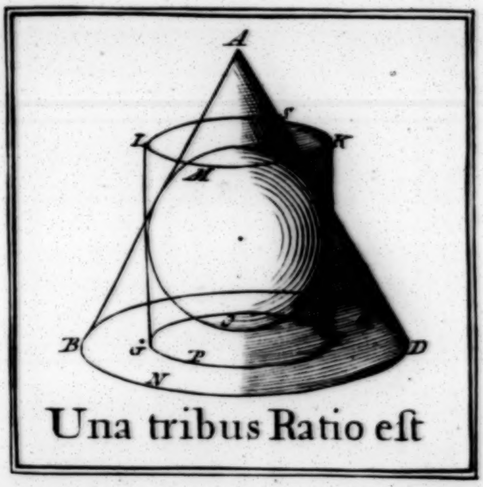
By ANDREW TACQUET,

Of the SOCIETY of JESUS;

And Demonſtrated in a more Eaſy and Compendious Way.

To which are added,

Some other agreeable Propoſitions, newly invented, by the ſame ANDREW TACQUET.



Una tribus Ratio eſt

see p. 67.
Prop. 45.

LONDON:

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TO THE

READER.

ALBEIT there have appear'd very many moſt excellent and admirable Men in the Mathematical Sciences; yet the chief Glory of all hath always, by a certain common Conſent, been given to Archimedes *of* Syracuſe. *Though indeed more there are who commend than who read him; more who admire than underſtand him. The Cauſes of which Neglect ſeem to be theſe, the Bulk and Scarceneſs of Copies, ſome Obſcurity of the Tranſlation, which is directly made out of the* Greek *Language, together with the Prolixity and Difficulty of his Demonſtrations. I judged therefore that it would be for the Profit of ſtudious Learners, if, after my Illuſtration of the Elements, I ſhould ſubjoin theſe* Theorems *which have been ſelected by me out of* Archimedes, *and demonſtrated in a much eaſier and briefer Way. Furthermore, I have ſelected thoſe, which bring along with them both more of Admiration and of Benefit; and have in my Demonſtration took ſuch a Method that, I hope, he who underſtands the Elements will, without any great Labour, comprehend theſe moſt excellent Inventions*

ventions of the Prince of Geometricians. I have alſo added at the End thirteen Propoſitions, and thereby enlarged the Doctrine of Archimedes *concerning the Sphere and Cylinder: where, amongſt other Things, I demonſtrate, that the ſeſquialteral Proportion is continued in the three Bodies, a Sphere, Cylinder, and equilateral Cone; both the latter being circumſcrib'd about the Sphere. Moreover, I have added divers Things here and there, amongſt which the 12th* Propoſition *and the* Corollaries *of* Prop. 14. *are the chief; and ſeveral* Scholiums. *Make uſe of theſe Diſcoveries whoſoever thou be'ſt, that art a Candidate of* Geometry; *and how much thou haſt improv'd in* Euclid, *make Proof of it in* Archimedes. *And, when thou perceiveſt thyſelf to be fix'd and rais'd upwards in the Contemplation of the moſt noble Truths, raiſe up thy Mind, while it is thus already lifted up from theſe lower Things yet higher, and direct it to that Truth which is Original, Eternal, Immenſe, and is no other than God; by the ineffable Viſion of whom, I truſt we ſhall hereafter be made eternally happy. Farewell.*

THEOREMS

Selected out of

ARCHIMEDES.

DEFINITIONS;

OR,

An Explication of certain Terms.

Fig. 23. Of the Table out of *Archimedes.*

LET there be a Circle BECG whoſe Centre is A, its Diameter BC, which let the right Line EG cut at right Angles, but not through the Centre, in D. Let there be drawn from the Centre the Radius's AE, AG. This being ſuppos'd,

Note, 1. That a Sector of a Sphere is that which is produced from the Sector of the Circle AECG, or AEBG, turn'd round about the Diameter BC.

2. That a Segment or Portion of a Sphere is that Part of it which is produc'd from the Segment of the Circle ECG or EBG turn'd round about the ſame Diameter BC.

3. The Vertex or Top of the ſpherical Portion EBG is the Extremity B of the unmov'd Diameter; the Baſis, the Circle deſcrib'd by EG; the Axis, that Part of the Diameter BD, which is intercepted betwixt the Top B, and D the Centre of the Baſe.

4. When I name the Superficies of a ſpherical Portion, or of a Body inſcribed in it, or of a Cone, I always underſtand it without the Baſe; and when I ſay the Superficies of a Cylinder, I mean likewiſe without the Baſes; unleſs the Word [whole] be adjoin'd to [Superficies]; for then the Baſes alſo are to be taken in.

Again, when I treat of Cylinders or Cones, I ſpeak of no other than right ones.

Axioms.

Fig. 1, 16. 1. THE Circuit of a Polygon inſcrib'd in a Circle is leſs than the Circumference of the Circle.

Fig. 1. 2. The Circuit of a Polygon deſcrib'd about a Circle is greater than the Circumference of the Circle.

Fig. 16. 3. And if a Polygon inſcrib'd in a Circle be turn'd about the Diameter (AE) together with the Circle, the Superficies of the Body produc'd by the Polygon will be leſs than the Superficies of the Sphere. And if a Polygon circumſcrib'd about the Circle be turn'd about the Diameter, together with the Circle, the Superficies of the Body produc'd by the Polygon will be greater than the Superficies of the Sphere.

Fig. 17. 4. In like manner the Circuit of a Polygon inſcrib'd in a Segment of a Circle (DAF) is leſs than the Circumference of the Segment. And if a Polygon inſcrib'd in the Segment be together with the Segment (A O) turned round, the Superficies of the Body produc'd by the Polygon will be leſs than the Superficies of the ſpherical Portion (D A F).

5. The Superficies of a Priſm inſcrib'd in a Cylinder is leſs than the Superficies of the Cylinder; but the Superficies of the Priſm which is circumſcrib'd is greater. Fig. 3, 6.

6. And the Superficies of a Pyramid inſcrib'd in a Cone is leſs than the Superficies of the Cone; but the Superficies of a circumſcrib'd Pyramid is greater. Fig. 4, 8.

PROPOSITIONS I, II.

ARE *not neceſſary.*

PROP. III. Theorem.

THE Circuits of Polygons circumſcrib'd about and inſcrib'd in a Circle do at laſt end in the Circumference of the Circle. In like manner the Polygons themſelves do at laſt end in the Circle.

If, to wit, the Arches being biſected without end, more and more Sides be circumſcrib'd about and inſcrib'd in the Circle. Fig. 1. Tab. *Archimedes.*

Part I. Let there be underſtood to be inſcrib'd in and deſcrib'd about a Circle regular Polygons; whether it be done ſo as is ſet down, *pr.* 12. *l.* 4. or as in the preſent Figure, the Thing will be the ſame. It is manifeſt (*per Coroll.* 1. *p.* 4. *l.* 6.) that FI is to CE (that is, the whole Circuit circumſcrib'd, unto the whole Circuit inſcrib'd) as I A is to CA. But IC, the Exceſs of the right Line IA above CA, becomes at length leſs than any given Line, if more and more Sides be underſtood to be infinitely circumſcrib'd and inſcrib'd; therefore alſo the Exceſs of the Circuit circumſcrib'd above that which is inſcrib'd will at length become leſs than any given Line. Therefore much more the Exceſs of the Circuit circumſcrib'd above the Circumference of the Circle will be leſs than any given one. In like manner, becauſe I have already ſhew'd the Defect of the Circuit inſcrib'd, whereby it falls ſhort of that which is circumſcrib'd, to be leſs than any given Line: therefore much more will the Defect of the Circuit inſcribed, whereby it falls ſhort of the Circumference of that Circle, become leſs than any given Line. The Circuits therefore, as well that which is inſcrib'd, as that which is circumſcrib'd, do at length (*Defin.* 6. *l.* 12.) end in the Circumference. Which was the firſt Part. To demonſtrate theſe Things further is not worth the while, ſeeing they are manifeſt enough.

Part II. Becauſe it hath already been ſhew'd that the Exceſs FI above the Side EC becomes at length leſs than any given (for FI is to EC, as IA to CA); therefore alſo the Exceſs of the Square of FI above the Square of EC will become at length leſs than any given. But as the Square of FI is to the Square of EC ſo (*per* 20. *l.* 6.) is the Polygon circumſcrib'd to that which is inſcrib'd. Therefore the Exceſs of the Polygon circumſcrib'd above that which is inſcrib'd will alſo become at length leſs than any given one. Therefore much more will the Exceſs of the Polygon circumſcrib'd above the Circle become at laſt leſs than any given one; and conſequently, the Defect alſo of the Polygon inſcrib'd, whereby it falls ſhort of the Circle, will at length become leſs than any given Defect. Therefore Polygons, as well inſcrib'd as circumſcrib'd, do at laſt *(Defin.* 6. *l.* 12.) end in the Circle. Which was the ſecond Part.

PROP. IV. Theorem.

A Regular [c] *Polygon (FINTR) circumſcrib'd about a Circle is equal to a Triangle, whoſe Baſe is the Circuit of the Polygon, and its Height the Radius of the Circle.*

[c] *Per Def.* 3. *l.* 4. Fig. 1.

And a regular Polygon inſcrib'd in a Circle is equal to a Triangle, which hath for its Baſe the Circuit of the Polygon, and for its Height the Perpendicular (AO) let down into one Side from the Centre.

Part I. The Radius A B drawn to the Point of Contact is *(per* 18. *l.* 3.) perpendicular to the Tangent I F. Wherefore if the right Lines A F, A I, A N, *&c.* being drawn, the Polygon be reſolv'd into Triangles, the Radius A B will be the common Altitude of all; and conſequently it is manifeſt that the Triangles are equal. Therefore a Triangle, which hath its Baſe equal to the Sum of the Sides F I, I N, N T, *&c.* and A B for its Altitude, will (as is manifeſt from 1. *l.* 6.) be equal to them all, that is, to the whole Polygon circumſcrib'd.

Part II. This may be concluded by the ſame Reaſoning as the other.

PROP. V. Theorem.

Fig. 2.

A Circle is equal to a Triangle, which hath for its Baſe the Circumference, and for its Height the Semidiameter of the Circle.

Regular Polygons circumſcrib'd about a Circle, and Triangles which have for their Baſes the Circuit of the Polygon, and for their Altitude the Radius of the Circle, are always (by the foregoing *Prop.*) equal. But Polygons circumſcrib'd infinitely about the Circle end in the Circle (by the 3d of this Book); and in like manner Triangles (as I will ſhew by and by) which have for their Baſe the Circuit of the circumſcribed Polygon, and for their Altitude the Radius A B, at laſt end in a Triangle, which hath the Circumference for its Baſe, and for its Altitude the Radius A B. Therefore (by the firſt) a Circle and a Triangle, which hath the Circumference for its Baſe, and the Radius for its Altitude, are equal.

But that Triangles contain'd under the Circuit of the Polygon, and the Radius of the Circle, end at laſt in a Triangle, which is contain'd under the Circumference and the Radius, I thus ſhew: Triangles under the Circuit of the circumſcribed Polygon and the Radius A B are to the Triangle which is under the Circumference and the Radius A B (by 1. *l.* 6.) as Baſe to Baſe, that is, as the Circuit of the Polygon to the Circumference; ſince this Triangle and the other have a common Altitude. But the Circuit of the Polygon (by the 3d) ends in the Circumference. Therefore the other Triangles end in this.

Corollaries.

1. FROM this and 41. *l.* 1. it is manifeſt, that a Rectangle under the Radius and Half the Circumference is equal to the Circle; that one under the Radius and the whole Circumference is double; that one under the whole Circumference and whole Diameter is quadruple thereto.

Fig. 5. l. 4. 2. A Circle is an inſcribed Square, as Half the Circumference (CDE) is to the Diameter; but to a Square circumſcribed, as the fourth Part of the Circumference is to the Diameter.

For the Rectangle under CDE, and the Radius CA or CF, that is (by the foregoing *Corollary)* the whole Circle, is to the Rectangle GFCE, to wit, the Rectangle under FG and CF (that is, to the inſcribed Square BCDE) as *(per* 1. *l.* 6.) CDE, Half the Circumference, is to FG or CE the Diameter; which was the firſt thing. And conſequently the Circle is to the Double of the Rectangle GFCE, (that is, to FH the circumſcribed Square) as CDE is to the Double of the Diameter CE, or as the Quadrant CD is to the Diameter CE.

Fig. 30. [3. *Of Figures which are of equal Circumferences the Circle is the moſt capacious. Let the Circumference of any Polygon whatſoever (as, for Inſtance, of a Square) EGHI be equal to the Circumference of the Circle. I ſay, that the Area of* *of the Circle is greater than that of the Polygon. For the Area of the Circle is equal to a Triangle, whoſe Baſe is the Circumference, and its Altitude the Semidiameter FA: And the Area of the Polygon is equal to a Triangle, whoſe Baſe is the Compaſs of the Polygon, which by the Hypotheſis is equal to the Circumference of the Circle, and which hath for its Altitude the Perpendicular FO, let down from the Centre of the Circle unto the Side of the Polygon; which, ſince it is always leſs than the Radius of the Circle, it is manifeſt that the Area of the Polygon is leſs than the Area of the Circle.* Q. E. D. *And, in like manner, amongſt all ſolid Figures contain'd under equal Surfaces, the Sphere may be demonſtrated to be the moſt capacious.*]

PROP. VI. Theorem.

THE Circumference of a Circle contains the Diameter leſs than thrice and one Seventh, (or $\frac{10}{70}$); and more than thrice and $\frac{10}{71}$.

For the Demonſtration of this Theorem, *Archimedes* aſſumes regular Polygons, one circumſcribed about a Circle, the other inſcrib'd, and both of them of 96 Sides. And then he ſhews, that the 96 inſcribed about a Circle do contain the Diameter leſs than thrice and one Seventh; and conſequently that the Circumference, which is leſs than them, doth alſo contain the Diameter leſs than thrice and one Seventh. But the 96 Sides inſcribed in the Circumference (and conſequently the Circumference alſo, which is greater than them) doth contain the Diameter more than three times $\frac{1}{7}$. But this Demonſtration is too long to be brought in in this Place. Nay, if we be minded to extend our Geometrical Reaſoning to Polygons of more Sides ſtill, we may contract the Limits even now ſet more and more without Limit, and ſo come nearer and nearer for ever to the true Proportion. This hath been perform'd by *Ludolph Ceulen*, *Grimberger*, *Metius*, *Snellius*, and others. The chief Proportions hitherto found I ſhall here ſubjoin.

[*However, ſince a Tangent of* 30 *Degrees multiplied by* 12 *gives the Circuit of a circumſcribed Hexagon*; *and a Sine of* 30 *Degrees multiplied by* 12 *gives the Circuit of an Hexagon which is inſcribed: Foraſmuch alſo as in like manner the Tangent of Half a Degree multiply'd by* 720 *yields the Circuit of a circumſcribed Polygon of* 360 *Sides*; *and the Sine of Half a Degree, the Circuit of an inſcribed Polygon of* 360 *Sides*; *and ſo on for ever: It will not be difficult to underſtand, by what Means many ſuch Numbers may be found, out of the now given Tables of Sines and Tangents.*]

The firſt Proportion, which is that of *Archimedes*, is thus:

The Diameter 7,

The Circumf. is 22; which is greater than the true.

The Diameter 71,

The Circumf. is 223; leſs than the true one.

The Proportions of 22 to 7, and 223 to 71, if they be reduced to a common Conſequent, (which is done after the ſame manner, in which Fractions are reduced to the ſame Denomination) will be thus: 1562 to 497, and 1561 to 497.

Therefore, the Diameter being ſuppos'd 497 Parts, the Circumference greater than the true one will be 1562, and the Circumference leſs than the true 1561.

Both of them therefore differ from the true, by a Quantity leſs than $\frac{1}{497}$ Part of the Diameter. But if the Proportions of 7 to 22, and 71 to 223, be reduced to a common Conſequent, there will ariſe the Proportions of 1561 to 4906, and of 1562 to 4906.

Therefore, the Circumference being ſuppos'd to be 4906 Parts, the Diameter leſs than the true will be 1561, the Diameter greater than the true 1562.

Both therefore differ from the true Diameter by Quantity leſs than $\frac{1}{4906}$ Part of the Circumference.

The Proportion deliver'd by *Metius* is much more accurate than this of *Archimedes.* According to this,

The Diameter is 113.
The Circumference 355.

Amongſt all Proportions conſiſting of ſmall Numbers, none comes nearer to the true one; for from this, the Diameter being ſuppos'd of 10,000,000 Parts, the Circumference comes to be of 31,415,929, which differs from the true one only in the firſt Figure 9, and this by an Exceſs but a little greater than two ten-millioneth Parts of the Diameter.

But more exact than both is that double Proportion of *Ludolphus* à *Ceulen*; the former of which conſiſts of 21 Figures, and the latter of 36.

The Diameter,
100,000,000,000,000,000,000.
The Circumf. greater than the true,
314,159265,358979,323847.
The Circumf. leſs than the true,
314,159265,358979,323846.

The Difference of both the Circumferences is one Particle of the Diameter denominated from a Number, which conſiſts of a Unity and 20 Cyphers; and conſequently, as well this as that differs from the true Circumference by a Quantity leſs than is the ſaid ſmall Part of the Diameter, to wit, one hundredth of a millioneth of a millioneth of a millioneth Part.

The Diameter,
100000,000000,000000,000000,000000,000000.
The Circumf. greater than the true,
314159,265358,979323,846264,338327,950289.
The Circumf. leſs than the true,
314159,265358,979323,846264,338327,950288.

The Difference of both the Circumferences, betwixt which is the true one, is that ſmall Part of the Diameter, denominated from a Number which conſiſts of Unity and 35 Cyphers; which ſmall Part bears a leſs Proportion to the whole Diameter, than one little Grain of Sand doth to the whole Globe of the Earth. For the whole Globe of the Earth doth not conſiſt of ſo many little Grains of Sand, as are the little Parts of the ſaid Sort which are contain'd in the Diameter.

It is needleſs to go any further. Neverthelеſs you may proceed infinitely, if you be minded to continue Geometrical Reaſoning, an expedite Method of which is deliver'd by *Snellius.*

[*The Circumference being ſuppos'd of*
1,000000,000000,000000,000000,000000,000000,000000 *Parts,*
The Diameter will be, as near as may be, of
0,318309,886183,790671,537767,926745,028724 *Parts.*]

Scholium.

THE moſt excellent Advantages of the Proportion now deliver'd are theſe which follow.

The

The Invention of the Diameter from the Circumference.

SET the greater Term of one of the Proportions, which have been now deliver'd, in the firſt place, the leſſer in the ſecond, the Circumference in the third; by theſe three Numbers let there be ſought by the golden Rule a fourth Proportional: That is the Diameter ſought.

As if the Circumference of the greateſt Circle of the Earth be ſuppos'd to contain 25000 *Engliſh* Miles of 5280 Feet each, and the Diameter be ſought, the Terms will ſtand thus:

355——113——25000——7854.

Multiply now the ſecond by the third, and divide the Product by the firſt, and there will ariſe 7854 Miles for the Diameter of the Globe of the Earth.

The Finding out of the Circumference from the Diameter.

LET the leſſer Term of one of the Proportions above deliver'd be ſet in the firſt place, the greater in the ſecond, the known Diameter in the third; and by theſe three Numbers let there be ſought a fourth Proportional: That will give the ſought Circumference.

As if the Diameter of the Globe of the Earth be ſuppos'd to contain 7854 *Engliſh* Miles, and the Circuit is ſought, the Terms will ſtand thus:

113——*355*——7854——25000.

Then multiply the ſecond by the third, and divide the Product by the firſt, there will ariſe

25000 Miles for the Circumference of the Globe of the Earth.

How little this Circumference exceeds the true one was ſaid above, to wit, by an Exceſs but a little greater than are two ten-millioneth Particles of the Earth's Diameter; that is, by 9 or 10 Feet. But, if we uſe the *Ludolphin* Proportion, even the former, the Terms whereof conſiſt of 21 Figures, there will be found a Circumference inſenſibly differing from the true, not only when the given Diameter is of 7854 Miles, ſuch as is the Diameter of the Earth; but alſo although the Diameter be ſuppos'd of an 100 Millions of thoſe Miles. For, this being ſuppos'd, there will ariſe a Circumference differing from the true one by a Quantity about one hundred-millioneth Part of a Foot. But, if to find out the Circumference of the Globe of the Earth we make uſe of the Proportion of *Archimedes*, the Interval of the two Circumferences, the one greater, the other leſs than the true one, will exceed 20 Miles. *Archimedes's* Proportion therefore is not to be uſed but in ſmall Meaſures; nay, it will always be expedient to uſe that of *Metius*, which both conſiſts of ſmall Terms, and is above 1000 Times more exact.

The Meaſuring of a Circle.

THE Semidiameter multiplied by half the Circumference produceth the Area of the Circle; as is manifeſt from *Corol.* 1. *Prop.* 5. of this Book.

As if the Semidiameter of the Earth, which contains 3926 Miles, be multiplied by half its Circumference, to wit, by 12500, there will ariſe 49,075000 Miles ſquare for the Area of the greateſt Circle of the Earth. The Difference of the circular Area thus found from the true is had, if the Difference of half this found Circumference from the true half Circumference be multiplied by the given Semidiameter; or the Difference of this Semidiameter from the true be multiplied by the given Semi-circumference.

The Menſuration of Cylinders and Cones.

I Put this here, becauſe it depends upon the Menſuration of a Circle. A Cylinder therefore, and any Priſm whatſoever, is produced from the Altitude multiplied by the Baſe: A Cone and Pyramid from the third Part of the Altitude multiplied by the Baſe; for they are third Parts of Cylinders and Priſms, having the ſame Baſe and Altitude with them, by 10, and 7. *l.* 12.

Let the Baſe of a Cylinder or Cone be of 50 ſquare Feet, and the Height of 100 Feet. Multiply 100 by 50, and there ariſe 5000 cubic Feet for the Solidity of the Cylinder. Multiply the third Part of the Altitude 100, which is $33\frac{1}{3}$ by 50, there ariſe $1666\frac{2}{3}$ cubic Feet for the Solidity of the Cone.

PROP. VII. Theorem.

Fig. 6. *l.* 12. *THE Circumferences of Circles have the ſame Proportion betwixt themſelves, which their Diameters have.*

For the Circuits of like Polygons, which may be inſcribed in a Circle without End, are always betwixt themſelves as the Diameters AF and IC by (*Corol. Pr.* 1. *l.* 12.) But theſe Circuits (by the third *Pr.* of this Book) end at length in the Circumference. Therefore their Circumferences alſo are betwixt themſelves as their Diameters. *Q. E. D.*

PROP. VIII. Theorem.

THE Superficies of a Priſm, as well that which is circumſcrib'd about, as that which is inſcrib'd in a Cylinder, is equal to a Rectangle whoſe Height is the Side of the Cylinder, but its Baſe equal to the Circuit of the Baſe of the Priſm.

Part I. The Superficies of the circumſcribed Priſm touches the Cylinder according to the Lines EA, NF, *&c.* which are the Sides of the Cylinder; but theſe (becauſe by the Hypothesis the Cylinder is a right one) are right to the Plain of the Baſe, and conſequently right alſo (by *Defin.* 3. *l.* 11.) to the Lines CG, GM, *&c.* But they are alſo equal betwixt themſelves. Therefore one Side of the Cylinder is the common Height of all the Rectangles CO, OM, MH, *&c.* Therefore the Superficies of the circumſcribed Priſm is equal (as is manifeſt from 1. *l.* 6.) to a Rectangle contain'd under the Circuit of the Baſe of the Priſm, and the Side of the Priſm or Cylinder. Fig. 3.

Part II. The Reaſon of this is the ſame. For the Side of the Cylinder is again the common Altitude of the Rectangles BDIK, KIQP, *&c.* which conſtitute the Superficies of the inſcribed Priſm.

PROP. IX. Theorem.

Fig. 4. THE *Superficies of a regular Pyramid circumſcrib'd about a right Cone is equal to a Triangle, which hath for its Baſe the Circumference (FHLD) of the pyramidal Baſe, but its Height the Side of the Cone (BG).*

And the Superficies of a regular Pyramid inſcribed in a right Cone is equal to a Triangle, which hath for its Baſe the Circumference of the pyramidal Baſe, but for its Height the Perpendicular (BO) let down from the Top unto a Side of the Baſe.

Part I. Let there be drawn unto the Contacts, G, K, M, the right Lines BG, BK, BM. Theſe will all be Sides of a right Cone, and conſequently equal. And, becauſe (by the Hypotheſis) the Axis BA is perpendicular to the Plain of the Baſe FKD, the Plain alſo GBA (*per* 18. *l.* 11.) will be perpendicular to the Plain FKD. But HG (*per* 18. *l.* 3.) is perpendicular to AG, the common Section of the Plains FKD and GBA. Therefore HG (as is gathered from *Defin.* 4. *l.* 11.) is alſo perpendicular to the Plain GBA. And conſequently is alſo perpendicular to BG. Therefore the Side GB of the Cone is the Height of the Triangle FBH. In the ſame manner the Side of the Cone will be the Height of the reſt HBL, LBD, *&c.* Therefore the Triangle comprehended under the Circumference

FHLD and the Side of the Cone is equal to the Superficies of a Pyramid circumſcribed, without the Baſe. Which was the firſt Part.

Part II. The Demonſtration of this Part is almoſt the ſame with that of the former.

PROP. X. Theorem.

THE *Superficies of a regular Priſm circumſcrib'd about a right Cylinder ends* (Defin. 6. l. 12.) *in the Superficies of the Cylinder; and the Superficies of a Pyramid circumſcrib'd about a right Cone ends in the Superficies of the Cone.*

Part I. The Superficies of regular Priſms, deſcribed about and inſcribed in a Cylinder without End, will have at laſt a Difference betwixt themſelves leſs than any which can be given. Much more therefore will the Superficies of a circumſcribed Priſm differ from the Superficies of the Cylinder, which is in the middle between the inſcribed and circumſcribed Superficies, by a Difference leſs than any given one whatſoever; that is, *(Def. 6. l. 12.)* will end in the cylindrical Superficies, whilſt it continually exceeds it leſs and leſs. Fig. 3.

Part II. This may be ſhewed in the ſame manner from the 9th and 3d of this. Fig. 4.

In the Figures there are only exhibited the Halves of the Cylinder and Cone, leſt a Multitude of Lines ſhould breed Confuſion. But the Cylinder and Cone are to be conceiv'd in the Mind entire, and as having thus circumſcribed Priſms and Pyramids encompaſſing them. For thus it more clearly appears that plain Surfaces circumſcribed are greater, according to the 3d *Axiom.*

A Lemma *to the following* Propoſition.

Fig. 7. LET AB, CD, EF, be proportional, and let KB be half AB, and EG, double EF; KB, CD, EG, will alſo be proportional.

The right Line KB is to AB as EF is to EG. Therefore the Rectangle KB, EG (*per* 16. *l.* 6.) is equal to the Rectangle AB, EF. But this (by 17. *l.* 6.) is equal to the Square of CD. Therefore alſo the Rectangle KB, EG, is equal to the Square of CD. Therefore (by 17. *l.* 6.) KB, CD, EG are proportional.

PROP. XI. Theorem.

Fig. 5, 6. *A Circle, whoſe Radius (GH) is a mean Proportional betwixt the Side of a right Cylinder (BC) and the Diameter of the Baſe (BD) is equal to the cylindrical Superficies.*

Let the regular and conſequently like Polygons, NM, RS, be underſtood to be circumſcribed about the Circles ABN, GPH; and upon the Polygon NM let a Priſm be conceived to be erected, with which the Cylinder is circumſcribed. Becauſe BD, GH, BC are by the Hypotheſis proportional, AD alſo (or AN) GH, and the Double of BC will by the *Lemma* be proportional. Now the Triangle contain'd under AN and the Circuit of the Polygon MN is equal to the Polygon circumſcribed NM (by the 4th of this Book): and the Rectangle under BC, or EF, and the ſame Circuit NM (that is, as is manifeſt from 41. *l.* 1. the Triangle under the Circuit NM, and the Double of BC) is equal (by the 8th of this Book) to the Superficies of a Priſm circumſcrib'd about the Cylinder. But a Triangle under the Circuit NM and AN is to the Triangle under the Circuit NM, and the Double of BC (by 1. *l.* 6.) as AN is to the Double of BC. Therefore the Polygon NM alſo is to the Superficies of a Priſm circumſcribed about a Cylinder, as AN is to the Double of BC. But becauſe I have already ſhew'd AN, GH, and the Double of BC to be proportional, the Proportion of AN to the Double of BC is (by *Defin.* 10. *l.* 5.) duplicate to the Proportion of AN to GH. Therefore the Polygon NM hath to the Superficies of the Priſm a Proportion duplicate to the Proportion of AN to GH. But the Polygon NM hath alſo to the Polygon like to it GRQS a Proportion duplicate to the Proportion of AN to GH, as is eaſily gathered out of 1. *l.* 12. Therefore the Polygon NM hath the ſame Proportion to the Superficies of the Priſm, which it hath to the Polygon GRQS; which conſequently is equal to the Superficies of the Priſm. In the ſame manner, I might ſhew that the priſmatic Superficies, which are circumſcriptible infinitely about the Cylinder, are always equal to the Polygons which may be circumſcribed infinitely about the Circle GPH. Wherefore ſeeing both the priſmatic Superficies (by the 10th of this) end in the Surface of the Cylinder, and the Polygons in the Circle GPA (by the 3d of this) the Superficies of the Cylinder alſo will be equal to the Circle GPH. *Q. E. D.*

From this admirable Theorem, a Circle is preſented which is equal to a cylindrical Superficies.

Corollaries.

Fig. 5, 6. *THE Superficies of a right Cylinder is equal to a Rectangle contain'd under the Side (BC) and the Circumference of the Baſe.*

The double of BC (as hath been ſhew'd above) is to GH, as GH is to BA, or AN; that is (by the 7th of this) as the Circumference P is to the Circumference BN. Therefore the Triangle under the firſt, to wit, the double of BC; and the fourth, to wit, the Circumference BN, is equal to a Triangle under the ſecond GH, and the third, to wit, the Circumference P (as appears from 16. *l.* 6.) But the Triangle under GH and the Circumference P is (by the 5th of this) equal to the Circle GPH, that is (by the 11th of this) to the cylindrical Superficies. Therefore alſo the Triangle under the Double of BC and the Circumference BN (that is, as appears from 41. *l.* 1. the Rectangle which is under BC and the Circumference BN) will be equal to the cylindrical Superficies. *Q. E. D.*

From this *Corollary* it is manifeſt, that the Properties of Rectangles are common to them with right cylindrical Superficies. Therefore let this be *Corollary* 2.

Fig. 20, 21. *l.* 12. 2. The cylindrical Superficies (BM, QN) which are of the ſame Height, are betwixt themſelves as the Diameters of their Baſes) BF, QR).

For the Rectangles under the Circumferences (CL, SE) and the equal right Lines FM, RN, to which (by *Corol.* 1.) the cylindrical Superficies are equal, are betwixt themſelves (by 1. *l.* 6.) as the Baſes, to wit, the Circumferences CL, SE; that is, as the Diameters BF, QR (by the 7th of this).

3. The cylindrical Superficies (CI, AR) which have equal Baſes are betwixt themſelves, as their Altitudes (TI, BR).

For the Rectangles contain'd under the equal Circumferences GH, MQ, and the Sides TI, BR, to which (by *Corol.* 1.) the cylindrical Surfaces are equal, are betwixt themſelves (by 1. *l.* 6.) as TI, BR. Fig. 23, 24. *l.* 12.

4. Like cylindrical Surfaces (BM, RI) have betwixt themſelves a Proportion duplicate to that, which (BF, QR) the Diameters of the Baſes have. **Fig. 20, 21. *l.* 12.**

Seeing the Cylinders are ſuppos'd to be like, MF will be to IQ (by *Defin.* 4. *l.* 12.) as BF is to QR; that is, (by the 7th of this) as the Circumference CL to the Circumference SE. Wherefore the Rectangles alſo which are contain'd under the Circumferences CL, SE, and the Sides MF, IQ, will be like; and conſequently they will have betwixt themſelves (by 20. *l.* 6.) a Proportion duplicate to that which MF hath to IQ; that is, BF to QR. Therefore the cylindrical Surfaces alſo have the ſame.

5. Cylindrical Surfaces (BM, RI) have betwixt themſelves a Proportion compounded of the Proportions of the Sides (FM, IQ), and the Diameters of the Baſes (BF, QR) as is manifeſt from 23. *l.* 6. and the 7th of this. **The ſame Figure.**

6. If

Fig. 24, 25. *l.* 12. 6. If cylindrical Surfaces (A R, F D) be equal; as the Diameter (A B) is to the Diameter F N) ſo reciprocally (by 14. *l.* 6.) the Altitude (F H) will be to the Altitude (K B); and converſly.

7. Laſtly, from the ſame 1ſt *Corol.* is had the Meaſure of a cylindrical Superficies; to wit, if the Circumference of the Baſe be multiplied by the Altitude. As if the Altitude be of 20 Feet, the Circumference of the Baſe of 6; multiply 20 by 6, there ariſe 120 ſquare Feet for the cylindrical Superficies.

PROP. XII. Theorem.

Fig. 5, 6, of this. *THE Superficies of a right Cylinder is to the Baſe (ABN) as the Side of the Cylinder (CB) is to (BO) the fourth Part of the Diameter of the Baſe.*

Let G H be a mean Proportional betwixt B C the Height, and B D the Diameter of the Baſe, and conſequently (by *Lemma* before *Prop.* 11. of this) a mean Proportional betwixt A N and the Double of B C. The Circle G P H of the Radius G H is (by the 11th of this) equal to the curve cylindrical Superficies C D. But the Circle G P H hath to the Baſe of the Cylinder A B N a Proportion duplicate (by 2. *l.* 12.) to the Proportion of G H to A N; that is, the ſame which the Double of B C hath to the Radius B A (by the Hypotheſis, and *Def.* 10. *l.* 5.) that is, the ſame which B C hath to B O, the fourth Part of the Diameter. Therefore the cylindrical Superficies alſo is to the Baſe A B N, as B C is to B O, the fourth Part of the Diameter. *Q. E. D.*

Corollary.

Corollary.

THE Superficies of a Cylinder, which hath its Side equal to the Diameter of its Baſe, is fourfold of the Baſe. But, if the Side be a fourth Part of the Diameter of the Baſe, the Superficies of the Cylinder will be equal to the Baſe. Both theſe are manifeſt from the Propoſition.

PROP. XIII. Theorem.

A Circle, whoſe Radius (O L) is a mean Proportional betwixt the Side (BC) of a right Cone, and the Radius of the Baſe (A C) is equal to the conical Superficies. Fig. 8, 9.

Let regular Polygons E F, I N, be underſtood to be circumſcrib'd about the Circles A C G, O P L, and a Pyramid circumſcrib'd about the Cone to be erected upon the Polygon E F.

Becauſe, by the Hypotheſis, A C or A G is to O L, as O L is to B C, the Proportion of A G to BC will *(Defin.* 10. *l.* 5.) be duplicate to the Proportion of A G to O L. But, as A G is to B C, ſo is the Triangle under A G and the Circuit E F to the Triangle under B C and the ſame Circuit E F. Therefore the Proportion of the Triangle under A G and the Circuit E F, to the Triangle under B C and the ſame Circuit, is alſo duplicate to the Proportion of A G to O L. But the Triangle under A G and the Circuit E F is equal to the Polygon E F (by the 4th of this): And the Triangle under B C and the ſame Circuit E F (by the 9th of this) is equal to the Superficies of the circumſcribed Pyramid. Therefore the Proportion of the Polygon EF to the Superficies of the Pyramid is alſo duplicate to the Proportion of AG to OL. But the Proportion of the Polygon EF to the Polygon IN, which is by the Conſtruction like to it, is *(per* 1. *l.* 12.*)* alſo duplicate to the Proportion of AG to OL. Therefore the Polygon EF hath the ſame Proportion to the Superficies of the Pyramid, and to the Polygon IN, which conſequently are equal. In the ſame manner I might ſhew, that the Superficies of Pyramids, which may be circumſcribed about a Cone infinitely more and more, are always equal to Polygons which may be circumſcribed infinitely about the Circle OPL. Wherefore, ſeeing both the Surfaces of Pyramids (by the 10th of this) do at laſt end in the Surface of the Cone, and Polygons (by the 3d of this) in the Circle OPL, the Superficies of the Cone and the Circle OPL ſhall likewiſe be equal. *Q. E. D.*

From this excellent Theorem a Circle is found, which is equal to a conical Surface.

Corollaries.

Fig. 8, 9. 1. THE Superficies of a right Cone is equal to a Triangle comprehended under the Side of the Cone (BC) and the Circumference of the Baſe (CG.)

Let OL the Radius be a mean Proportional betwixt AC and BC. Then, becauſe (by the 7th of this) the Circumference CG is to the Circumference P, as the Radius AG is to the Radius OL; that is, by the Hypotheſis, as OL is to BC; the Triangle under the firſt, to wit, the Circumference CG and under the fourth BC (as appears from 16. *l.* 6.) will be equal to the Triangle under the ſecond; to wit, the Circumference P, and the third OL; that is (by the 5th of this) to the Circle OPL; that is, to the conical Superficies (by the 13th of this) BCD. *Q. E. D.*

From this *Corollary* it appears that conical Surfaces have the ſame Properties with Triangles. And ſo it follows,

2. That the conical Superficies (BAF, QXR) having their Sides (BA, QX) equal, are betwixt themſelves as the Baſes of their Diameters (BF, QR.) And, Fig. 20, 21. *l.* 12.

3. Thoſe which have equal Baſes CFT, AZB, are betwixt themſelves as their Sides (CF, AZ.) And, Fig. 23, 24. *l.* 12.

4. Thoſe conical Superficies (BAF, QZR) which are like, have betwixt themſelves a Proportion duplicate to that which is betwixt the Diameters of the Baſes. And, Fig. 20, 21. *l.* 12.

5. All conical Superficies whatſoever have betwixt themſelves a Proportion, which is compounded of the Proportions of the Sides (BA, QZ) and of the Diameters (BF, QR) which are in the Baſes. And, *The ſame Figure.*

6. Thoſe which are equal have their Sides and the Diameters of the Baſes reciprocally proportional; and thoſe which have them ſo are equal.

All which is demonſtrated from *Coroll.* 1. as above we deduced the *Corollaries* concerning the cylindrical Surface out of the firſt *Corollary* there.

7. Laſtly, we may meaſure a right conical Surface, if we multiply the Side FC by Half the Circumference of the Baſe. As if the Side be Fig. 25. *l.* 12.

be of 5 Feet, the Circumference of the Baſe of 20; multiply 5 by 10, and there will ariſe 50 ſquare Feet for the conical Superficies. The Demonſtration is manifeſt from the ſame firſt *Corollary.*

PROP. XIV. Theorem.

Fig. 8, 9, of this.

THE *Superficies of a right Cone is to the Baſe, as the Side (BC) is to (AC) the Radius of the Baſe.*

Between the Side BC and AC the Radius of the Baſe, let OL be a mean Proportional. Therefore the Proportion of BC to AC is duplicate to (*Defin.* 10. *l.* 5.) the Proportion of OL to AC. Now (by the 13th of this) a Circle of the Radius OL is equal to the conical Superficies CBD. But the Proportion of this to ACG the Baſe of the Cone is (by 2. *l.* 12.) duplicate to the Proportion of OL to AC; and conſequently is the ſame with the Proportion of BC to AC. Therefore the Proportion of the conical Superficies CBD is to the Baſe ACG, as BC is to AC. *Q. E. D.*

Corollaries.

Fig. 27.

1. THE *Superficies of a right Cone, produc'd by an equilateral Triangle turn'd about the Perpendicular (KA) is double to the Baſe (QT).*

For the Side KB is equal to BD, and conſequently double to the Half AB, which is the Radius of the Baſe.

2. *The Superficies of a Cone produc'd by a right-angled equicrural Triangle (E B D) is to the Baſe, as in a Square the Diameter is to the Side.* Fig. 24.

For, the Perpendicular B A being drawn, the right Angle B (by 26. *l.* 1.) is biſected, and conſequently A B D is half right. But A D B is alſo an half right Angle (by *Coroll.* 11. *Pr.* 32. *l.* 1.) Therefore D A, B A, are (by 6. *l.* 1.) equal; and conſequently B D is the Diameter of the Square A K, whereof A D is the Side. Now the ſame A D is the Semidiameter of the Baſe P T, ſeeing the Perpendicular A B (by 26. *l.* 1.) biſects E D. From which and this 14th the *Corollary* is manifeſt.

3. *The Superficies of the right Cylinder (G K) is to the Superficies of the right Cone (G B N) as the Side of the Cylinder is to Half the Side of the Cone.* Fig. 24.

For the Superficies of the Cone G B N is to the Baſe M I, as the Side B N is to Q N the Semidiameter of the Baſe (by the 14th of this); that is, as Half the Side B N is to the fourth Part of the Diameter G N. But the Baſe M I (by the 12th of this) is to the Superficies of the Cylinder G K, as the fourth Part of the Diameter is to N K, the Side of the Cylinder. By Equality of Proportion therefore the conical Superficies G B N is to the cylindrical Superficies G K, as Half the Side of the Cone is to N K, the Side of the Cylinder. *Q. E. D.*

A LEMMA *to what follows.*

Fig. 10. IN a Triangle, as NPV, let there be drawn QD parallel to NV.

I ſay, that the Rectangle under PN and NV is equal to the Rectangle under PQ, QD, together with the Rectangle under NQ and the two NV, QD put together.

Draw NA perpendicular to the Side NP, and equal to NV; and, the Rectangle NO being compleated, let the Diameter PA be drawn. Then from Q let there be drawn QE parallel to NA, which may cut PA in B. Through B let CF be drawn parallel to NP. Becauſe AN is equal to NV, it is manifeſt that QB alſo is equal to QD (from *Coroll.* 1. *Pr.* 4. *l.* 6.) Therefore the Rectangle ON is the Rectangle PNV, and FQ is PQD. It remains that we prove that the Rectangles OB, EC, BN, are equal to the Rectangle under NQ, and the two NA, BQ; that is, to the Rectangle under NQ, and the two Lines NV, QD. But that is manifeſt; for the Rectangle under NQ and NA, QB, is equal *(per* 1. *l.* 2.) to theſe three Rectangles; that under NQ and CA (that is, the Space EC) and that under NQ and NC (that is, the Space BN) and that under NQ and QB, that is again the Space BN, and conſequently the Space OB, which *(per* 43. *l.* 1.) is equal to BN. The Propoſition therefore is manifeſt.

PROP. XV. Theorem.

IF a right Cone be cut by the Plain QSR parallel to NZO; I ſay, that the Circle GHM, whoſe Radius GH is a Mean betwixt Part of the Side NQ and QD, NV (the Radius's of the Circles QSR, NZO) taken together, is equal to the conical Surface intercepted betwixt the parallel Circles QSR, NZO. Fig. 11, 12.

Let GF be the Mean betwixt PN and NV. Likewiſe let GK be the Mean betwixt PQ and QD; and let there be deſcribed the Circles GFL, GKT. This (by the 13th of this) will be equal to the conic Superficies QPR, and the other to the Superficies NOP. The Rectangle PNV (by the *Lemma)* is equal to the Rectangle PQD, together with the Rectangle under NQ and NV QD taken together. But becauſe (by the Conſtruction) GF is a mean Proportional betwixt PN, NV; the Rectangle PNV is equal to the Square of GF (by 17. *l.* 6.) And becauſe GK is (by the Conſtruction) a Mean betwixt PQ, QD, the Rectangle (by 17. *l.* 6.) PQD is equal to the Square of GK: And becauſe GH by the Hypotheſis is a Mean betwixt QN and QD, NV, taken together, the Rectangle (by 17. *l.* 6.) under QN, and QD, NV, taken together, is equal to the Square of GH. Therefore the Square of GF is alſo equal to the Square of GH, and to that of GK. Therefore ſeeing Circles are betwixt themſelves (by 2. *l.* 12.) as the Squares of their Radius's, the Circle GLF will alſo be equal to the two Circles **G K T, G H M**, taken together. But (by the 13th of this) the Circle **G L F** is equal to the conical Superficies **N P O**. Therefore the conical Superficies **N P O** is alſo equal to the two Circles **G K T** and **G H M**. But **Q P R** one Part of the Superficies **N P O** is (by the ſame) equal to the Circle **G K T**. Therefore the remaining Part, which is comprehended betwixt the parallel Circles **Z Z, S S**, is equal to the Circle **G H M**. *Q. E. D.*

A LEMMA *to what follows.*

Fig. 13. RIGHT Lines (**B H, C G**) which in the Circle intercept equal Arches (**B C, H G**) are parallel.

For let **C H** be drawn. Becauſe the Arches **B C, H G** are by the Hypotheſis equal, the alternate Angles alſo (by 29. *l.* 3.) **B H C, G C H** will be equal. Therefore (by 28. *l.* 1.) **B H** and **C G** are parallel. *Q. E. D.*

PROP. XVI. Theorem.

Fig. 13. L*ET there be inſcribed in a Circle a regular Figure of an even Number of Sides, and let it be equilateral; let E B be drawn from the Extremity of the Diameter unto B, the End of the Side next to the Diameter; and let the right Lines B H, C G, D F, join the Angles which are equally diſtant from A.*

I ſay, that the Rectangle contain'd under the Diameter A E, and the Subtenſe E B, is equal to the Rectangle of one Side of the inſcri-

bed Figure AB, or BC, &c. *and of all the joining Lines BH, CG, DF, taken together.*

Draw CH, DG: Becauſe BH, CG, DF intercept (by 26. *l.* 3.) equal Arches, BC, HG; CD, GF; theſe Lines (by the *Lemma*) will be parallel. By the ſame Argument BA, CH, DG, EF are parallel. All the Triangles therefore (by 27 and 15. *l.* 1.) BAK, KHL, LCM, MGN, NDO, OFE, are equiangular. Therefore (by 4. *l.* 6.) as BK is to KA, ſo is HK to KL; and as HK is to KL, ſo is CM to ML; and as CM to ML, ſo is GM to MN; and as GM is to MN, ſo is DO to ON; and as DO is to ON, ſo is FO to OE. Therefore (by 12. *l.* 5.) as one of the Antecedents, BK, is to one of the Conſequents KA; ſo all the Antecedents, BK, KH, CM, MG, DO, OF (that is, all the joining Lines BH, CG, DF) are to all the Conſequents AK, KL, LM, MN, NO, OE (that is, to the Diameter AE.) But (by 8. *l.* 6.) as BK is to AK, ſo is EB to BA. Therefore, as all theſe together, BH, CG, DF are to AE, ſo is EB to BA. Therefore (by 16. *l.* 6.) the Rectangle under BA on one Part, and all the joining Lines BH, CG, DE, on the other, is equal to the Rectangle which is under AE and EB. *Q. E. D.*

PROP. XVII. Theorem.

LET there be inſcribed in DAF a Segment of a Circle, whoſe Baſe DF is perpendicular to the Diameter AOE, a Figure Fig. 14.

gure equilateral, and of an even Number of Sides; and let there be drawn, as in the foregoing, the Line E B.

I ſay, that the Rectangle comprehended under E B, and A O Part of the Diameter, is equal to the Rectangle which is under one Side of the inſcribed Figure, and all the joining Lines B H, C G, &c. *taken together with D O Half the Baſe.*

The Demonſtration is the ſame with that of the foregoing.

LEMMA I. *to what follows.*

Fig. 13. LET there be inſcribed in the greateſt Circle of a Sphere a regular Figure, which hath its Sides meaſured by the Number Four, and ſtands about the Axis A E; which Axis remaining unmov'd, let the Circle be turn'd round together with the Figure:

I ſay, that there will be inſcrib'd in the Sphere a Body contained under conical Superficies.

It is manifeſt (ſee *Defin.* 2. *l.* 12.) that B A, H A, likewiſe D E, F E, deſcribe entire Superficies of right Cones. Then, becauſe the Lines C B, G H, and G F, C D, being produced, do concur on both Sides in the ſame Point of the Diameter A E, which is in like manner to be drawn out, and cuts the joining Lines perpendicularly; it is alſo manifeſt, that the ſaid Lines C B, G H, *&c.* do deſcribe Parts of right conical Surfaces, which are intercepted betwixt the parallel Circles, which the Tops of the Angles B, C, D, deſign in the ſpherical Superficies.

LEMMA 2.

LET DAF be the greateſt Section of a Segment of a Sphere, whoſe Axis is A O. Let there be inſcribed in this a Figure having all the Sides equal, the Baſe excepted, and let it be turn'd round about the Axis A O.

I ſay, that a Body contain'd under conical Superficies will be inſcrib'd in the ſpherical Segment.

This is proved as the foregoing *Lemma.*

PROP. XVIII. Theorem.

LET the ſame things be ſuppos'd which were in the firſt Lemma; *and let the right Line (E B) be drawn from the Extremity of the Diameter unto the End of the Side next to the Diameter.* Fig. 13.

I ſay, that a Circle, the Square of whoſe Radius (I) is equal to the Rectangle A E B contain'd under the Diameter A E, and the Subtenſe E B, is equal to all the conical Surfaces inſcrib'd in the Sphere.

That is, a Circle whoſe Radius (I) is a mean Proportional betwixt A E and E B.

Becauſe the right Lines B H, C G, D F, are equal to the right Lines B K, C M, D O, taken twice; (by 1. *l.* 2.) the Rectangle under one Side of the Figure inſcrib'd in the greater Circle (to wit, under A B, or B C, or C D, or D E) and under all the joining Lines together B H, C G, D F, is equal to the Rectangle under A B and B K, together with that which is under B C and the Compound of BK and CM, together with that which is under CD and the Compound of CM and DO, together with that which is under DE and DO; for ſo each of the Lines BK, CM and DO, are taken twice. But (by the 16th of this) the Rectangle under AB, and all the joining Lines BH, CG, DF, taken together, is equal to the Rectangle AEB; that is (by the Hypotheſis) to the Square of I. Therefore the Square of I is equal to the Rectangles under AB and BK, and under BC and the Compound of BK and CM, under CD and the Compound of CM and DO, and under DE and DO. Now let P be a mean Proportional betwixt AB and BK, and Q a Mean betwixt BC and the Compound of BK and CM, and R a Mean betwixt CD and the Compound of CM, DO; S a Mean betwixt DE and DO. The Squares therefore of P, Q, R, S (by 17. *l.* 6.) are equal to the aboveſaid Rectangles. Wherefore, ſeeing I have already ſhew'd the Square of I to be equal to the ſame Rectangles, it muſt alſo be equal to the Squares of P, Q, R, S together. Seeing therefore (by 2. *l.* 12.) Circles are betwixt themſelves as the Squares of their Radius's, the Circle deſcribed of the Radius I will alſo be equal to all the Circles together, whoſe Radius's are P, Q, R, S, (as is manifeſt from 22. *l.* 6.) But the Circles of the Radius's P and S are (by the 13th of this) equal to the conical Superficies which the Sides AB, ED have produc'd, foraſmuch as P is a mean Proportional betwixt AB the Side of the Cone, and BK the Radius of the Baſe; and S is a mean Proportional betwixt ED and DO, and the Circle of the Radius Q is (by the 15th of this) equal to that Segment of a conical Superficies, which is intercepted betwixt the two parallel Circles of the Diameters C G, B H; becauſe Q is a Mean betwixt B C and the Compound of B K, C M: And for the ſame Cauſe the Circle of the Radius R is equal to a Segment of a conical Surface, which is intercepted betwixt the parallel Circles of the Diameters C G, D F. Therefore the Circle deſcribed from the Radius I is equal to the conical Surfaces inſcribed in the Sphere taken all together. *Q. E. D.*

PROP. XIX. Theorem.

LET the ſame things be ſuppos'd which were in the 2d Lemma, *and let the right Line E B be drawn from the Extremity of the Diameter (A E) to the End of A B the Side next to the Diameter.* Fig. 14.

I ſay, that a Circle, whoſe Radius is a mean Proportional betwixt (E B) and (A O) the Axis of the Segment, is equal to all the conical Superficies inſcribed in the ſpherical Segment D A F.

The Demonſtration is altogether the ſame with that of the foregoing, only for *Prop.* 16. let *Prop.* 17. be cited.

PROP. XX. Theorem.

COnical Superficies inſcrib'd in a Sphere do at length end in the Superficies of the Sphere. Fig. 15.

Let there be given a Superficies as ſmall as you will, as X. It is manifeſt, that within the ſpherical Superficies A C E G there may be given ſome other concentrical thereto, which falls ſhort of this by a Quantity leſs than X. Let A C E G, D P L M, be the greateſt Circles of both, as cut with a Plain through the Centre. Let there be drawn the Diameter A D E, and in D let N Q touch it. If the Arch A E be biſected in C, and again the Remainder be biſected, and ſo on, there will be left at laſt the Arch A B (as is manifeſt of itſelf) leſs than the Arch A N. If to this Arch the right Line A B be ſubtended, it is manifeſt that it will not reach to the Circumference P D M L, and that it will be a Side of an equilateral Figure, of an even Number of Sides inſcribed in the Circle C A G E, no Side whereof reacheth unto the Circumference P D M L. Wherefore, if all be turn'd round about the Diameter A E, it is manifeſt that there will be inſcribed in the exterior ſpherical Surface conical Surfaces, which include the ſpherical Surface, which is concentrical to the other, and conſequently (by *Axiom* 3. of this) are greater. Becauſe therefore the ſpherical Surface D P L M falls ſhort of the ſpherical Surface A C E G, by a Quantity leſs than the given one X; much more will the conical Surfaces fall ſhort of the ſaid ſpherical Surface A C E G by a Quantity leſs than the given one X, and (by *Defin.* 6. *l.* 12.) conſequently will end in the Superficies A C E G. *Q. E. D.*

PROP.

PROP. XXI. Theorem.

Conical Superficies inſcribed in a ſpherical Segment D A F end in the ſpherical Superficies of the Segment itſelf. Fig. 17.

This may be demonſtrated by the ſame Reaſoning as the foregoing was.

PROP. XXII. Theorem.

IT was demonſtrated Prop. 18. *that a Circle, whoſe Radius is a mean Proportional betwixt the Diameter A E and the right Line E B, which is drawn from the Extremity of the Diameter unto the End of the Side A B next to the Diameter, is equal to all the conic Superficies inſcrib'd in the Sphere.* Fig. 16.

I ſay, that this Circle (ſee Def. 6. l. 12.*) ends at length in a Circle, whoſe Radius is A E the Diameter of the Sphere.*

For, if more and more Sides be infinitely inſcribed in the greateſt Circle (which then being turn'd round about A E produce conical Superficies) it is manifeſt that the Side A B becomes at length leſs than any given right Line, and conſequently that the Subtenſe E B approaches to the Diameter A E within a Diſtance leſs alſo than any given one; from whence it comes to paſs that the Difference A E, D E, becomes likewiſe leſs than any given one. Therefore much more ſhall the mean Proportional betwixt A E, B E, which is always greater than B E, differ from AE at length by a Defect leſs than any given one. Therefore the Circle alſo, whoſe Semidiameter is a Mean betwixt AE and BE, will at length differ from a Circle whoſe Semidiameter is AE, by a Defect leſs than any given whatſoever; that is, will end (*Def.* 6. *l.* 12.) in it. *Q. E. D.*

This, which is clear enough of itſelf, there is no need to demonſtrate more operoſely.

PROP. XXIII. Theorem.

Fig. 17. IT *was demonſtrated* Prop. 19. *that a Circle, whoſe Radius is a mean Proportional betwixt* E B *and the Axis of the Segment* A O, *is equal to all the conical Superficies inſcrib'd in the ſpherical Portion* DAF.

I ſay, that this Circle ends in a Circle, whoſe Radius is the right Line AD drawn from the Vertex of the Segment unto the Circumference of the Circle D *Q* F N, *which is the Baſe of the Segment.*

For becauſe it now appears from the foregoing Demonſtration that EB doth at length end in AE; it will alſo be manifeſt that the mean Proportional betwixt EB and AO doth at length end in the mean Proportional betwixt AE and AO, that is (by *Corol.* 2. *p.* 8. *l.* 6.) in AD itſelf. It is therefore manifeſt that the Circle alſo, whoſe Radius is a mean Proportional betwixt EB and AO, doth end in the Circle of the Radius AD. *Q. E. D.*

A Lemma

A Lemma *to the following* Propoſition.

IF the Diameter of one Circle be double to the Diameter of another, the one Circle will be fourfold to the other.

This is manifeſt from *Prop.* 2. *l.* 12. and *Defin.* 10. *l.* 5.

PROP. XXIV. Theorem.

THE Superficies of every Sphere is fourfold of the greateſt Circle of the ſame Sphere. Fig. 16.

This moſt noble Theorem of *Archimedes* we ſhall from what goes before expeditiouſly demonſtrate in this manner:

Let an ordinate Figure, the Sides whereof are meaſured by the Number four, be underſtood to be inſcrib'd in the greateſt Circle of a Sphere about the Diameter A E. Let this Figure be turn'd round about A E, and ſo produce conical Surfaces inſcrib'd in the ſpherical Surface, and let E B be drawn. It hath been demonſtrated above (18. of this) that all conical Surfaces inſcribed in a Sphere are equal to the Circle, the Square of the Radius whereof is equal to the Rectangle A E B, that is, whoſe Radius is a mean Proportional betwixt A E and E B. And this will always happen, although the Inſcription be infinitely continued. Wherefore ſeeing the inſcrib'd conical Surfaces (by 20. of this) will at length end in the ſpherical Surface, and the Circle whoſe Radius is a Mean betwixt A E and E B will at length end (by 22. of this) in the Circle whoſe Radius is A E;

the ſpherical Surface itſelf alſo (by 2. of this) will be equal to the Circle of the Radius A E, that is, (by the foregoing *Lemma)* to four Times the greateſt Circle A C E G. *Q. E. D.*

He that ſhall read *Archimedes* will find that the Way here uſed in demonſtrating this moſt noble Theorem is much ſhorter and clearer than that of *Archimedes.*

Corollary.

FROM this admirable Theorem, whereby *Archimedes* hath purchas'd to himſelf an immortal Name amongſt the Geometricians, a Circle equal to a ſpherical Surface is obtain'd; that to wit, whoſe Semidiameter is the Diameter of a Sphere, or whoſe Diameter is double to the Sphere's Diameter.

Scholium.

WE are now well provided for the Meaſuring of a ſpherical Surface, the chief among[illegible]ll curve ones. And it is perform'd theſe two Ways:

1. Let the greateſt Circle of the Sphere be meaſured (according to *Schol. Prop.* 6. of this) and let it be multiplied by 4. As, if the greateſt Circle of the Orb of the Earth be found to contain 49,075,000 ſquare Miles, then according to this 196,325,000 ſquare Miles are contain'd in the whole ſpherical Surface.

2. The Diameter of a Sphere multiplied by the Circumference of the greateſt Circle gives you the ſpherical Superficies. According to which, if the Earth's Diameter conſiſt of 7,853 Miles, and conſequently the Circumference of

the greateſt Circle conſiſts of 25,000, the whole ſpherical Surface will be in the ſame Miles 196,325,000; for 7,853 × 25,000 = 196,325,000.

The Demonſtration appears from *Coroll.* 1. *Prop.* 5. of this; for a Rectangle under the Diameter of a Sphere, and the Circumference of the greateſt Circle, is according to that *Corollary* fourfold of the greateſt Circle.

PROP. XXV. Theorem.

THE Surface of any ſpherical Portion whatever (as D A F*) is equal to a Circle, whoſe Radius is the right Line (*A D*) drawn from the Vertex of the Portion to the Circumference of the Circle (*D Q F N*) which is the Baſis of the Segment.* Fig. 17.

Let a Figure equilateral and of an even Number of Sides, the Baſe being ſet aſide, be imagin'd to be inſcrib'd in the Section of the greateſt Portion about the Axis A O; this Figure being turn'd round about A O will inſcribe conical Surfaces in the Portion. Let the right Line E B be drawn alſo as above (in 18 and 19 of this). All the conical Surfaces now inſcribed are equal (by the 19th of this) to the Circle whoſe Radius is a mean Proportional betwixt E B and the Axis of the Segment A O. And this will always happen if the Inſcription be infinitely continued. Wherefore ſeeing both the conical Surfaces inſcrib'd in the Segment end at length (by 21 of this) in the ſpherical Surface of the Segment, and the Circle whoſe Radius is a Mean betwixt E B and A O ends (by 23.) in the Circle of the Radius A D;

the ſpherical Surface of the Portion alſo D A F (by 2.) will be equal to the Circle of the Radius A D. *Q. E. D.*

This is another of the more noble Inventions of *Archimedes*, which as the former we have demonſtrated in a much ſhorter and clearer Way than he did.

PROP. XXVI. Theorem.

Fig. 18. *THE Superficies of a right Cylinder circumſcrib'd about the Sphere (as the Cylinder H P S V) is equal to the Surface of the Sphere.*

And if a Cylinder and Sphere be cut by Plains perpendicular to the Axis (B G); each Segment of the cylindrical Surface will be equal to each Segment of the Sphere.

Part I. Becauſe the Side H P of the Cylinder is (by the Hypotheſis) equal to P S the Diameter of the Baſe ; the cylindrical Surface H S will be (by *Coroll. p.* 12. of this) fourfold of the Baſe ; that is, of the greateſt Circle of the Sphere inſcrib'd in the Cylinder ; of which ſeeing (by 24 of this) the ſpherical Surface itſelf is alſo fourfold, this will be equal to the cylindrical Surface. *Q. E. D.*

Part II. Let the right Lines B O, G O, be drawn. Becauſe the Angle BOG (by 31. *l.* 3.) is right, as being the Angle in the Semicircle, and OC falls perpendicular from it upon BG ; BO (by *Corol.* 2. *p.* 8. *l.* 6.) will be a mean Proportional betwixt GB and BC, that is, betwixt I T and H I. Therefore the Circle of the Radius BO (by 11. of this) will be equal to the cylindrical Surface HT. But the ſame Circle is alſo (by the foregoing) equal to the Segment of the ſpherical Surface OBK. Therefore the cylindrical Surface HT and the ſpherical OBK are equal.

Then, becauſe it is ſhew'd in the ſame manner that the cylindrical Surface HX is equal to the ſpherical QBR, the remaining cylindrical Surface IX will be equal to the remaining ſpherical Surface QOKR, which is intercepted betwixt two parallel Circles.

And from theſe the Propoſition is manifeſt of all Segments.

[Coroll. *Hence the Superficies of a Cylinder circumſcrib'd about a Circle is double to the Baſes.*]

PROP. XXVII. Theorem.

THE Segments of a ſpherical Surface divided by parallel Circles have that Proportion amongſt themſelves, which the Segments (BC, CD, DA, AE, EF, FG) of that Diameter (BG) which is perpendicular to the parallel Circles have amongſt themſelves. Fig. 18.

It follows from the foregoing. For by that the Segments of the ſpherical Surface OBK, QOKR, MQRN, &c. are equal to the Cylindrical HT, IX, LN, &c. But theſe (by 13. *l.* 12.) have the ſame Proportion betwixt themſelves which the Segments of the Axis BC, CD, DA, &c. have Therefore thoſe alſo have the ſame Proportion. *Q. E. D.*

Scholium.

Scholium.

FROM hence the Proportion of Zones and Climates betwixt themſelves becomes known. For they are to one another as the Segments of the Axis, which are known from the Table of Sines.

From the ſame alſo we learn to meaſure the Segments of a ſpherical Surface. For becauſe both the whole Surface of the Sphere is known from *Schol. Prop.* 24. and the Proportion of the Segments, the ſame as that of the Parts of the Axis, is alſo given; it is manifeſt that each of the Segments become known.

Now both the foregoing, and all the reſt of the Theorems which follow, are altogether ſingular and admirable, and well worthy that thoſe who are ſtudious of Geometry ſhould give all Diligence to underſtand them.

A Lemma *to the following.*

Fig. 19. IF a Plain (QN) touch a Sphere in (O) a right Line (A O) from the Centre to the Contact is perpendicular to the Plain.

Let QN the touching Plain and the Sphere be cut through the Contact with two Plains, which in the Sphere may produce the Circles O G, O D, but in the Plain QN the right Lines C O, I O, which ſhall touch the Circles in O. Therefore by 18. *l.* 3. AO is perpendicular to both IO and CO, and conſequently by 4. *l.* 11. perpendicular to the Plain QN. *Q. E. D.*

PROP.

PROP. XXVIII. Theorem.

Every Sphere is equal to a Cone (ZO) whoſe Altitude (KO) is equal to the Radius of the Sphere; and the Baſe (Z) equal to the Superficies of the Sphere. Fig. 20, 21, 22.

Let ſome polyedral Body be underſtood to be circumſcribed about the Sphere, and let the ſolid Angles thereof be cut off by new Plains touching the Sphere. Which being done there will ariſe another polyedral Body containing the Sphere, but leſs than the former, and conſiſting of more Angles, and having a Surface compounded of more tangent Plains in Number but leſs in Magnitude. If the ſolid Angles of this Polyedrum be again cut off by new tangent Plains, and the Angles of the third Polyedrum thence ariſing likewiſe, and ſo on for ever; it will come to paſs at length that both the Polyedrum will exceed the Sphere by a Solid leſs than any given one whatſoever; and the Surface thereof compounded of tangent Plains (which, as I ſaid, are endleſly leſs in Magnitude, and more in Number than they were before) will exceed the ſpherical Surface alſo by a Plain leſs than any given one whatever. Both which Things, although they might be demonſtrated, yet, becauſe they are of themſelves manifeſt enough, I ſhall for Brevity's ſake take for granted. Theſe Things being thus ſtated, we proceed.

The Polyedrum now deſign'd is compounded of Pyramids, the common Top whereof is the Centre of the Sphere, and the Baſes are tangent Plains, which conſtitute the Surface of the Polyedrum.

lyedrum. And becauſe the right Lines, drawn from the Centre A unto the Contacts of each of the Plains, are (by the foregoing *Lemma*) perpendicular to each of the Plains; therefore the Height of all the Pyramids, whereof the Polyedrum conſiſts, will be equal; to wit, AB the Radius of the Sphere. If therefore the Plain X be ſuppoſed equal to the Surface of the Polyedrum itſelf, and upon it there be erected a Pyramid at the Height MN, which is alſo equal to the Radius of the Sphere; it is manifeſt (by 6. *l.* 12.) that all the aboveſaid Pyramids, that is, the whole Polyedrum, are equal to the Pyramid XN. After the ſame manner all the reſt of the Polyedrums containing the Sphere, which from the perpetual Abſciſſion of the ſolid Angles will ariſe one after another infinitely, are always equal to the Pyramids (repreſented by XN) the Altitudes whereof MN are the Radius of the Sphere; but the Baſes (X) equal to the Surfaces of Polyedrums encompaſſing the Sphere. Wherefore, ſeeing at length both the Polyedrums (as I ſaid above) do end in a Sphere, and the Pyramids (XN) as I will ſhew by and by, do end in the Cone ZO, the Sphere alſo (by 1. of this) will be equal to the Cone. *Q. E. D.*

But that the Pyramids XN end in a Cone, I thus ſhew: The Surfaces of Polyedrums end in the Surface of the Sphere, as it was taken for granted above. But the Baſes X of the Pyramids XN are always ſuppos'd equal to the Surfaces of the Polyedrums; and Z, the Baſe of the Cone ZO, is by the Hypotheſis equal to the Surface of the Sphere; therefore the Baſes X alſo will end in the Baſe Z; and conſequently ſeeing the Pyramids XN be to the Cone, which by the Hypotheſis is of equal Height (by *Corol. Prop.* 11. *l.* 12.) as the Baſe X is to the Baſe Z, the Pyramids alſo will end in the Cone.

The Demonſtration of this Propoſition and the following is altogether diverſe from that which *Archimedes* made uſe of, which indeed is very ſubtile and ingenious, but prolix and difficult; to which there are premis'd two Poſitions that are manifeſt, and eleven Propoſitions, beſides others not a few, on which they depend. But the Theorem itſelf, as propounded by *Archimedes*, is thus; every Sphere is fourfold of a Cone, which hath a Baſe equal to the greateſt Circle of the Sphere, and its Altitude equal to the Radius.

Scholium.

FROM this noble Theorem is deduc'd the Menſuration of the moſt noble of ſolid Figures. For if the ſixth Part of the Diameter, or the third Part of the Semidiameter be multiplied by the Surface of the Sphere, already known by *Schol. Prop.* 24. there will ariſe the Solidity of the Sphere.

Suppoſe the Superficies of the Earth be found to contain 196,325,000 ſquare Miles, and let the third Part of the Semidiameter conſiſt of 1309 ſuch Miles. Multiply the two Numbers together, the Product 256,989,425,000 will be the Number of the cubic Miles of the Earth's Solidity.

For ſeeing a Sphere (by this *Prop.*) is equal to a Cone whoſe Altitude is the Radius of the Sphere, and its Baſe the Surface of the ſame Sphere, and the Solidity of the Cone (by *Schol. Prop.* 6. of this) is produc'd from the third

Part of the Altitude (that is, of the Radius of the Sphere) multiplied by the Baſe (that is, the Surface of the Sphere) the Sphere's Solidity alſo is obtain'd from the 3d Part of the Radius multiplied into the Superficies.

PROP. XXIX. Theorem.

Fig. 23. *EVery Sector of a Sphere is equal to a Cone, whoſe Altitude is the Radius of the Sphere, and the Baſe the ſpherical Superficies of the Sector.*

Firſt, let the Sector AECG be leſs than an Hemiſphere. Let a right-lin'd polyedral Body be underſtood to be circumſcrib'd about the Sector. Now, if all the remaining Ratiocination be carried on after the ſame manner as was done in the foregoing, the Thing ſought will be concluded in the ſame manner. This Thing alone will require to be ſhew'd, upon which indeed the whole Reaſoning depends; to wit, that the Superficies of the Polyedrum, which is compounded of Plains on every Side, touching the Surface of the Sphere ECG, is greater than the Surface ECG. Which is done thus: Let another equal and like Surface be conceiv'd to be ſet to the Surface ECG encompaſs'd with touching Plains in the very ſame manner as the other is. There will now (by *Axiom* 3. of this) the whole Surface compounded of Plains be greater than the whole ſpherical Surface. Therefore half the Surface compounded of Plains will alſo be greater than half the ſpherical Surface ECG. Then let the Sector AEBG be greater than an Hemiſphere. Both Sectors taken together are (by the foregoing) equal to a Cone a Cone whoſe Height is the Radius of the Sphere, and its Baſis the whole Superficies; that is (by 11. *l.* 12.) to two Cones which have the ſame Height, but have their Baſes equal to the Segments of the ſpherical Superficies ECG, EBG. But one of the Sectors AECG, that which is leſs than an Hemiſphere, is by Part 1. equal to a Cone, whoſe Altitude is the Radius, and its Baſe the Surface ECG. Therefore the other Sector AEBG is equal to the other Cone whoſe Height is the Radius, and its Baſe the remaining ſpherical Surface EBG. *Q. E. D.*

Corollary.

SEEING (by 25. of this) the Superficies ECG is equal to the Circle of the Radius CG, and the Superficies EBG equal to the Circle of the Radius BG; the Sectors AECG and AEBG will be equal to Cones whoſe Altitude is the Radius of the Sphere, and their Baſes Circles of the Radius's CG and BG.

Scholium.

FROM theſe Things is deduc'd the Meaſuring both of Sectors and Segments of Spheres; of Sectors (as appears from *Schol. Prop.* 6. of this) if the third Part of the Radius be multiplied by the ſpherical Surface of the Sectors, which is already known from *Schol. Prop.* 27. or by the Circle of the Radius CG or BG; and of Segments, if the Cone EAG be meaſured, and be taken away from the Sector, if it be leſs than an Hemiſphere; but added thereto if it be greater. Fig. 23.

Fig. 18. The Segment (MQRN) which lies betwixt two Circles, whether parallel or not parallel, is meaſured; if the Segments QBR and MBN, already known, be ſubtracted one out of the other.

PROP. XXX. Theorem.

Fig. 24. *AN Hemiſphere (EOBD) is double to the Cone (EBD) which hath the ſame Baſe and Altitude with itſelf.*

The Cone whoſe Baſis is the hemiſpherical Superficies EOBD, and its Altitude the Radius AB, is to the Cone EBD (by 11. *l.* 12.) as Baſe is to Baſe; that is, as the hemiſpherical Surface EOBD is to the greateſt Circle PT. Therefore, ſeeing the hemiſpherical Superficies EOBD is double to the greateſt Circle (by 24. of this) the Cone alſo which hath the Superficies EOBD for its Baſe, and the Radius AB for its Altitude, is double to the Cone EBD. But (by 28. of this) the Hemiſphere is equal to a Cone which hath the Radius for its Altitude, and the hemiſpherical Superficies for its Baſe. Therefore the Hemiſphere is alſo double to the Cone EBD. *Q. E. D.*

PROP. XXXI. Theorem.

Fig 25. *LET a Sphere be divided into two Segments ILBG, ISKG, by the Plain IQGT which doth not paſs through the Centre A; and let the Diameter BOK be perpendicular to the cutting Plain.*

As the Altitude O B *of the Segment* ILBG *is to the Radius of the Sphere* AB*: ſo let* O K, *the Altitude of the other Segment, be made to the other Line* K N.

In like manner, As O K, *the Altitude of the Segment* ISKG, *is to the Radius* AK *or* AB, *ſo let the Altitude* O B *of the other Segment be made to the other Line* B D. *Which Things being ſuppos'd, I ſay,*

1. *The Cones* ING *and* IDG *whoſe Altitudes are* ON, OD, *and* IQGT *their common Baſe, are equal to the ſpherical Segments.*

2. *There is the ſame Proportion of the Segments as there is of the right Lines* D O, NO.

3. *The Segment* ISKG *is to the greateſt Cone* IKG *inſcribed in it, as* NO *is to* KO; *and the Segment* ILBG *is to the greateſt Cone* IBG *inſcrib'd in it, as* DO *is to* BO.

Part I. Let the Sphere and Cones be cut by a Plain through the Diameter B K. There will be produced in the Sphere the greateſt Circle BLKG, and in the Cones the Triangles BIG, IKG. And becauſe BOK the Diameter is (by the Hypotheſis) perpendicular to the Circle QT, IOB (by *Def.* 3. *l.* 11.) will be a right Angle. The Angle in the Semicircle is alſo a right one (by 31. *l.* 3.) Becauſe therefore in the Triangle BIK there is drawn from the right Angle IO perpendicular to the Baſe BK, BI will be to IO, as (by 8. *l.* 6.) BK to KI. Therefore the duplicate Proportion of BI to

I O is equal to the duplicate Proportion of B K to K I; that is (becauſe B K, K I, K O (by *Corol.* 2. *Pr.* 8. *l.* 6.) are three Proportionals) equal to the Proportion of B K to K O.

Then, becauſe O B is (by the Hypotheſis) to B D, as O K is to the Radius A B; by Inverſion it will be always thus: D B is to B O, as A B to O K; and by Permutation thus: D B is to B A, as B O to O K; and by Compounding thus: D A is to B A, as B K is to O K. Becauſe therefore I have already ſhew'd the Proportion of B K to O K to be duplicate to the Proportion of B I to I O, and conſequently (by 2. *l.* 12.) equal to the Proportion betwixt the Circles deſcrib'd by the Radius's B I, I O; D A will alſo be to B A, as the Circle of the Radius B I to the Circle of the Radius I O. Therefore the Cone under the Altitude D A, and for the Baſe, the Circle of the Radius I O; that is, the Circle Q T is equal to the Cone under the Altitude B A (by 15. *l.* 12.) which hath for its Baſe the Circle of the Radius B I; that is (by *Corol.* 29. of this) the ſpherical Sector A I B G. Wherefore, if the ſame Cone I A G be added, as well to the Sector A I B G, as to the Cone under D A, and the Circle Q T, the Wholes will be equal; to wit, the ſpherical Segment I L B G will be equal to two Cones; whereof one is that which is under the Baſe Q T and the Altitude D A, and the other I A G is under the ſame Baſe Q T and the Altitude O A. But theſe two Cones (by 14. *l.* 12.) make up the Cone I D G. Therefore the Segment I L B G will be equal to the Cone I D G. *Q. E. D.*

By the ſame Reaſoning the Segment I S K G will be equal to the Cone I N G, with this only

Change, that the Cone I A G, which before was added, be now taken away.

Part II. This is manifeſt out of the firſt. For the Cones I D G and I N G are betwixt themſelves (by *Pr.* 14. *l.* 12.) as are D O and N O. Therefore the Segments alſo I L B G, I S K G, equal to thoſe Cones, are betwixt themſelves, as the right Lines D O, N O.

Part III. This likewiſe is manifeſt from the firſt. For the Cone I D G is to the Cone I B G (by the ſame) as D O is to B O. Therefore the Segment alſo I L B G, which is equal to the Cone I D G, is to the Cone I B G, as D O is to B O.

Scholium.

FROM the firſt Part of this Propoſition there ariſes another Way of meaſuring ſpherical Segments, and that a very eaſy one, if, to wit, the Cones I D G, I N G, be meaſured; which will be done, if the third Parts of the right Lines D O, N O, be drawn into the Circle Q T.

PROP. XXXII. Theorem.

A Right Cylinder (G K) is both in Solidity and the whole Superficies to the Sphere about which it is circumſcrib'd, as 3 *to* 2. Fig. 24.

Let B Q be the common Axis of the Sphere and Cylinder, and E B D the greateſt Cone inſcrib'd in the Hemiſphere E O B D. Becauſe the Cylinder E K (Half of G K) is (by 10. *l.* 12.)

l. 12.) treble to the Cone EBD, while the Hemiſphere is double to the ſame Cone (by 30 of this) it is manifeſt that the Cylinder EK is to the Hemiſphere as 3 to 2. Therefore alſo the whole Cylinder GK is to the whole Sphere QEBD as 3 to 2. Which was the firſt Part.

Then, becauſe the Side of the Cylinder KN is equal to GN the Diameter of the Baſe, its Superficies without the Baſes will be fourfold (by *Corol. Pr.* 12. of this) of the Baſe MI, and conſequently taken together with the Baſes, that is, the whole Superficies of the Cylinder, will be ſixfold of the Baſe MI, which is equal to the greateſt Circle of the Sphere. But the Superficies of the Sphere is fourfold of that greateſt Circle. Therefore the whole Superficies of the Cylinder GK is to the Superficies of the Sphere, as 6 to 4, or as 3 to 2. Which was the other Part.

Therefore a Cylinder is both in Solidity and the whole Superficies to the Sphere about which it is circumſcrib'd, as 3 to 2. *Q. E. D.*

Scholium.

IT is an Argument what a great Value *Archimedes* puts upon this Theorem, that he would have a Sphere inſcrib'd in a Cylinder ſet upon his Tomb. And perhaps, amongſt ſo many other famous Diſcoveries, this chiefly and above all others pleas'd him, for this Reaſon, to wit, becauſe there was one and the ſame rational Proportion both of Bodies, and of the Surfaces which contain them. We have demonſtrated a like Identity of Affections betwixt Rings, and the Surfaces of Rings, in the 4th Book of our Cylindrics and Annularies, *Prop.* 13, 14, 15. And another famous Example of the ſame hath alſo offer'd itſelf to me in the Sphere itſelf. For I have found, that like as a Sphere is to a right Cylinder which encompaſſeth it (which will neceſſarily be equilateral) as 2 is to 3, and this both in reſpect of Solidity and Surface; ſo likewiſe the Sphere hath, to an equilateral Cone encompaſſing it, that Proportion which 4 hath to 9; and this both in regard of Solidity and Superficies. From which this alſo follows, That the ſeſquialteral Proportion found by *Archimedes* in the Sphere and Cylinder is continued in three Solids, Sphere, Cylinder, and equilateral Cone. The Demonſtration of both which things, with ſome other Theorems of my own, in which the wonderful Nature of the Sphere will more appear, I ſhall ſubjoin in the thirteen following Propoſitions.

PROP. XXXIII. Theorem.

THE Superficies of a Sphere is double to the Superficies of a ſquare Cylinder inſcrib'd in the ſame Sphere. Fig. 26.

Let AKL be the Square inſcrib'd in the greateſt Circle of a Sphere, from which turned round there is deſcribed a ſquare Cylinder; and let AL be drawn as a Diameter common to the Cylinder and Sphere. Becauſe the Square AL is (by 47. *l.* 1.) equal to the equal Squares AK, KL, it will be double to one AK. Therefore alſo the Circle of the Diameter AL is (by 2. *l.* 12.) double to the Circle, whoſe Diameter is AK; to wit, to the Circle CN. But the Superficies of the Sphere is (by 24 of this) fourfold to the Circle, whoſe Diameter is

AL; for that is the greateſt Circle of the Sphere, ſeeing AL is the Diameter of the Sphere. Therefore the Superficies of the Sphere is eightfold of the Circle CN. But, becauſe LK, KA (by the Hypotheſis) are equal, the cylindrical Superficies ACL is (by *Corol. Pr.* 12. of this) quadruple of the Circle CN. Therefore, ſince the Superficies of the Sphere is eightfold of the ſame Circle, it will be double to the cylindrical Superficies. *Q. E. D.*

PROP. XXXIV. Theorem.

Fig. 26. *THE Superficies of a Sphere hath that Proportion to the whole Superficies of a ſquare Cylinder inſcrib'd in it, which* 4 *hath to* 3.

Let the ſame things be ſuppos'd which were in the foregoing Demonſtration. Becauſe by the Hypotheſis LK, the Side of the Cylinder, and AK, the Diameter of the Baſe thereof, are equal, the cylindrical Superficies CL will be quadruple (by *Corol. Prop.* 12. of this) to the Baſe CN, and conſequently the whole Superficies of the Cylinder is to both Baſes CN and SL, as 6 is to 2. But the Superficies of the Sphere is to both Baſes together CL, SL, as 8 is to 2; ſeeing in the foregoing it was ſhew'd, that it is to one Baſe as 8 to 1. Therefore the Superficies of the Sphere is to the cylindrical Superficies CL, as 8 is to 6, or 4 to 3. *Q. E. D.*

Corollary.

THE whole Superficies of a right Cylinder deſcrib'd about a Sphere is to the whole Superficies of an equilateral Cylinder inſcrib'd, as 2 is to 1. For the circumſcrib'd is to the ſpheric Superficies as 12 is to 8 (by 32 of this.) But the Spheric is to the inſcrib'd as 8 is to 6 by this preſent Propoſition. Therefore the circumſcrib'd is to the inſcrib'd, as 12 is to 6, or 2 to 1.

PROP. XXXV. Theorem.

Fig. 26, or 25.

A Portion of any ſpherical Superficies whatever (as ILBG*) hath the ſame Proportion to the Superficies of the greateſt inſcribed Cone, which (*BG*) the Side of the Cone hath to (*GO*) the Radius of the Baſe.*

Becauſe (by 25. of this) the Superficies of the Portion ILBG is equal to the Circle of the Radius BG; the Proportion thereof to QT, that is, to the Baſe of itſelf and of the Cone, will be duplicate to the Proportion (by 2. *l.* 12.) of BG to GO; that is (by 14. of this) of the Proportion of the conical Superficies IBG to the ſame Baſe QT. Therefore it is manifeſt (by *Def.* 10. *l.* 5.) that the Superficies ILBG is to the conical Superficies IBG, as the ſame conical Superficies IBG is to the Baſe QT. Wherefore, ſeeing the conical Superficies IBG is to the Baſe QT, as BG (by 14. of this) is to GO, the Superficies of the Portion will alſo be to the conical Superficies IBG inſcrib'd in it, as BG is to GO. *Q. E. D.*

PROP.

PROP. XXXVI. Theorem.

Fig. 24. *THE Superficies of the Hemiſphere (EOBD) hath that Proportion to (EBD) the Superficies of the greateſt right inſcribed Cone, which in a Square the Diameter hath to a Side; and that Proportion to the Superficies of a like Cone circumſcribed, as the Side in a Square hath to the Diameter.*

I. The Demonſtration of the firſt Part is manifeſt from the foregoing. For EOBD the Superficies of any Portion whatever, and conſequently of the Hemiſphere, is to the conical Superficies inſcrib'd, as BD is to DA. But BADK is a Square, whoſe Diameter is BD and the Side DA.

Fig. 6. *l*. 4. Part II. Let EBC be Half of the Square circumſcrib'd about the Circle (whoſe Centre is A); which EBC being turn'd about the Axis AB, let there from thence be produc'd a Cone circumſcribed about the Hemiſphere. Now, becauſe the Square EC is (by 47. *l*. 1.) double to the Square EB or GI, the Circle of the Diameter EC alſo is (by 2. *l*. 12.) double to the Circle whoſe Diameter is GI, that is, to the Circle HGDI. But (by 24. of this) the Superficies of the Hemiſphere included in the Cone EBC is double to the ſame Circle. Therefore the Circle of the Diameter EC is equal to the hemiſpherical Surface. Wherefore, ſeeing the conical Superficies EBC is (by 14. of this) to the Circle of the Diameter EC, to wit, to its own Baſe, as the Side BE is to EA the

Radius of the Baſe, it will be alſo to the hemiſpherical Superficies inſcribed in it, as BE is to EA; that is, as the Diameter in a Square is to a Side. *Q. E. D.*

PROP. XXXVII. Theorem.

A Sphere hath the ſame Proportion to a ſquare conical Rhombus circumſcribed about it both in reſpect of Solidity and Surface, which in a Square the Side hath to the Diameter.

The ſame Figure with Fig. 13 *l.* 5.

Let the Square EBCF be circumſcrib'd about HGDI, the greateſt Circle of a Sphere; from which Square, as turn'd round about the Axis BF, let a conical Rhombus encompaſſing the Sphere be produc'd.

As EB a Side of the Square (ſee *Fig.* 6. *l.* 4.) is to the Diameter EC, even ſo let S be made to R; (ſee *Fig.* 13. *l.* 5.) and let this Proportion be continued through four Terms, S, R, Q, O; the Proportion then of S to O will be triplicate to the Proportion of S to R; that is, of EB to EC; and the Proportion of O to R will be duplicate to the Proportion of O to Q, or of R to S; that is, of EC to EB; and conſequently (by 20. *l.* 6.) O is to R, as the Square of EC is to EB; from whence (by *Schol. Pr.* 6 and 7. *l.* 4.) O is double to R. Theſe things being thus ſettled, let the Sphere EBCF be underſtood to be circumſcribed about the conical Rhombus. Thus the Sphere HGDI will be to the Sphere EBCF (by 18. *l.* 12.) in the triplicate Proportion of the Diameter GI or EB to the Diameter EC; that is (as I have already ſhew'd) it will be as S to O. But the Sphere E B C F is to the conical Rhombus inſcribed in it (by 30. of this) as 2 is to 1; that is (as I have ſhew'd above) as O is to R. Therefore, by Equality of Proportion, the Sphere H G D I is to the ſame Rhombus, which is deſcribed about it, as S is to R; that is, as in a Square the Side E B is to the Diameter E C. Which was the firſt Part. Then, from the ſecond Part of the foregoing, it appears that the Superficies of the Hemiſphere is to the Superficies of the Cone E B C, and conſequently the Superficies of the whole Sphere is to the Superficies of the whole Rhombus E B C F, as in a Square the Side is to the Diameter. Therefore the Sphere, as well in Solidity as in Superficies, is to the ſquare Rhombus E B C F, as in a Square the Side is to the Diameter. *Q. E. D.*

PROP. XXXVIII. Theorem.

Fig. 27.

THE Superficies of the Portion (B G K D) which contains an equilateral Cone (B K D) is double to the Superficies of the ſame Cone.

This is manifeſt from 35. For the Superficies of the Portion B G K D is to the inſcrib'd conic Superficies (by 35. of this) as B K is to B A. But, becauſe the Cone B K D is ſuppos'd to be equilateral, K B is equal to B D, and conſequently double to B A. Therefore the Superficies B G K D is alſo double to the inſcribed conical Superficies. *Q. E. D.*

PROP.

PROP. XXXIX. Theorem.

THE Superficies of a Sphere is to the whole Superficies of an equilateral Cone inſcribed in it, as 16 *is to* 9. Fig. 27.

Let Z be the Centre of the Sphere, and BKD the equilateral Cone inſcribed, and KZAO the Axis common to the Sphere and Cone. If the Sphere and Cone be cut through this, there will be produced in the Sphere the greateſt Circle, and in the Cone the equilateral Triangle BKD, one Side whereof will be the Diameter of the Baſis of the Cone QT. And, becauſe the Axis of the Cone KA is perpendicular to the Baſe QT, BAK *(Def.* 3. *l.* 11.) will be a right Angle. Therefore the Square of BA is equal to the Rectangle KAO (*Coroll.* 1. *Pr.* 17. *l.* 6.) Now, becauſe the Side of the equilateral Triangle cuts off (*Coroll.* 5. *Pr.* 15. *l.* 4.) a fourth Part of the Axis AO, the Rectangle KAO, that is, the Square of BA, will be treble to the Square of AO (by 1. *l.* 6.) Wherefore, ſeeing the Square of the Radius ZO is (*Coroll.* 3. *Pr.* 1. *l.* 2.) quadruple of the Square of AO, the Square of the Radius ZO will be to the Square of the Radius BA, as 4 is to 3. Therefore the Circle OBKD is alſo (by 2. *l.* 12.) to the Circle QT, as 4 is to 3. Therefore four Circles OBKD, that is (by 24 of this) the whole ſpherical Superficies DG is to the Circle QT, as 16 is to 3. But *(Coroll.* 1. *Pr.* 14. of this) the Superficies of the equilateral Cone BKD is to the Circle QT, to wit, its own Baſe, as 2 is to 1, and conſequently the whole Superficies of the Cone BKD, that is, including its Baſe, is to the Baſe, to wit, the Circle QT, as 3 is to 1, or 9 to 3. Therefore, ſeeing I have ſhew'd that the Superficies of a Sphere is to the ſame Circle as 16 is to 3, the Superficies alſo of the Sphere DG will be to the whole Superficies of the equilateral Cone as 16 is to 9. *Q. E. D.*

Or otherwiſe thus:

BEcauſe by *Corol.* 5. *Prop.* 15. *l.* 4. the Side BD of the equilateral Triangle cuts off a 4th Part of the Axis AO, the ſpherical Superficies BOD will be a 4th Part by 27 of this, and conſequently the Superficies BGDK three 4th Parts of the Superficies of the whole Sphere. Wherefore, if the whole Superficies be ſuppos'd to be 16, the Superficies BGKD will be 12. But (by the foregoing) the Superficies BGKD is double to the cónical Superficies BKD, and conſequently is to it as 12 to 6. Therefore the whole Superficies of the Sphere is to the conical BKD as 16 is to 6. Then, becauſe the Superficies of the Cone BKD (as being equilateral) is (by *Corol.* 1. *Pr.* 14. of this) double to the Baſe QT, it is manifeſt that the conical Superficies BKD (to wit, without the Baſe) is to the whole Superficies of the Cone as 2 is to 3; that is, as 6 to 9. Therefore, by Equality of Proportion, the whole Superficies of the Sphere is to the whole Superficies of the equilateral Cone inſcrib'd, as 16 to 9. *Q. E. D.*

PROP. XL. Theorem.

Fig. 28. *THE Superficies of a Sphere bears the Proportion to the whole Superficies of an equilateral Cone circumſcrib'd about it, that* 4 *doth to* 9.

Let there be circumſcrib'd about the greateſt Circle of a Sphere B P M the equilateral Triangle D O F, by which, as turn'd round about the Axis O A B, let there be produc'd an equilateral Cone circumſcrib'd about the Sphere. And let there alſo be circumſcrib'd about the equilateral Triangle D O F the Circle N D L O F, which, as is manifeſt, is concentrical to the former; and let the Axis O A B be produc'd to N. Becauſe B N is a fourth Part of the Axis O N, as is manifeſt (from *Coroll.* 5. *Pr.* 15. *l.* 4.) O N is double to K B. Wherefore, the Proportion betwixt Circles being duplicate (by 2. *l.* 12.) of the Proportion of the Diameters, the Circle B P M will be to the Circle N D L O F, as 1 to 4. But it hath already been ſhew'd in the firſt foregoing Demonſtration, that the Circle N D L O F is to the Circle Q T, the Baſe of the equilateral Cone inſcrib'd in the Sphere F L, as 4 is to 3. Therefore, by Equality of Proportion, the Circle B P M is to the Circle Q T, as 1 is to 3. But the whole Surface of the Cone D O F is (by *Cor.* 1. *Pr.* 14. of this) treble to Q T. Therefore the whole Superficies of the Cone is ninefold of the Circle B P M. Wherefore, ſeeing the Superficies of the Sphere T P is quadruple (by 24. of this) of the ſame Circle B P M, the whole Superficies of the equilateral Cone D O F is to the Superficies of the Sphere to which it is circumſcrib'd, as 9 is to 4. *Q. E. D.*

[Coroll. 1. *From this Demonſtration it is manifeſt, that the Axis B O of an equilateral Cone circumſcrib'd about a Sphere is one and an half of the Diameter of the Sphere E K, or as 3 to 2.*

2. That QT the Baſe of the Cone DOF *is alſo one and an half of both Baſes of the Cylinder circumſcrib'd about the ſame Sphere. For QT is to* BPM *as* 3 *to* 1. *Therefore QT is to* BPM *twice as* 3 *is to* 2.

3. That the Superficies of the Cone DOF *is one and an half of the Superficies of the equilateral Cylinder circumſcrib'd about the ſame Sphere. For that*[a] *is double to QT, while this is quadruple to* BPM[b]. *Therefore the conical Superficies will be to the Cylindrical, as twice* 3 *to four times* 1; *that is, as* 6 to 4, *or as* 3 *to* 2.]

[a] *Per Corol.* 1. *p.* 14 of this.

[b] 24 and 26 of this.

PROP. XLI. Theorem.

Fig. 28. *THE whole Superficies of an equilateral* Cone *circumſcrib'd about a Sphere, is quadruple to the whole Superficies of a* Cone *inſcribed in the ſame Sphere.*

By the foregoing, the whole Superficies of the equilateral Cone DOF circumſcrib'd is to the Superficies of the Sphere, as 9 to 4; and the Superficies of the Sphere is to the whole Superficies of the inſcribed Cone SKT, as 16 to 9 (by 39. of this.) Therefore, by Perturbation of Equality of Proportion, the whole Superficies of the circumſcribed equilateral Cone is to the whole Superficies of the equilateral inſcribed, as 16 is to 4, or as 4 to 1. *Q. E. D.*

PROP. XLII. Theorem.

Fig. 29. *A Sphere hath that Proportion to* BKC, *an equilateral* Cone *inſcribed in it, which* 32 *hath to* 9.

Let the Sphere and Cone be cut by a Plain paſſing through the common Axis KO, producing in the Sphere the greateſt Circle OFKI, and in the Cone the equilateral Triangle BKC. Then, a Plain being drawn through the Centre A perpendicular to OK, let the Hemiſphere FGKI be cut off, in which let the greateſt Cone FKI be underſtood to be inſcribed. Now, becauſe (by *Cor.* 5. *Pr.* 15. *l.* 4.) the Side BC of the equilateral Triangle cuts off a fourth Part of the Axis OK, PK will be to AK as 3 to 2, that is, as 9 to 6. But the Baſe QT is to the Circle OFKI, that is, to the Baſe ND, as 3 to 4, that is, as 6 to 8, as appears from what was demonſtrated *Pr.* 39. Wherefore, ſeeing the Proportion of the Cone BKC to the Cone FKI is (by *Schol.* 2. *Pr.* 15. *l.* 12.) compounded of the Proportion of the Altitude PK to the Altitude AK (that is, of the Proportion of 9 to 6) and of the Proportion of the Baſe QT to the Baſe ND (that is, of the Proportion of 6 to 8) the Cone BKC will be to the Cone FKI, as 9 to 8. Wherefore, ſeeing (by 30. of this) the Sphere CG is quadruple of the Cone FKI, the equilateral Cone BKC will be to the Sphere CG, as 9 to 32. *Q. E. D.*

PROP. XLIII. Theorem.

AN equilateral Cone circumſcrib'd about a Sphere, is eightfold of an equilateral Cone inſcrib'd in the ſame Sphere. Fig. 28.

Let SKT and DOF be the equilateral Cones inſcrib'd and circumſcrib'd, and let OKB be the common Axis. Then let as well both the Cones as the Sphere be cut by a Plain paſſing through the Axis; their Sections will be two equilateral Triangles, and the greateſt Circle BPM. About the Triangle DOF likewiſe let there be underſtood to be deſcrib'd the Circle NDOF, but let the Axis OKB be produc'd unto N. Now, becauſe the Side DF of the equilateral Triangle doth (by *Corol.* 5. *Pr.* 15. *l.* 4.) cut off NB the fourth Part of the Axis ON, it is manifeſt that ON is double to BK. In like manner, becauſe the Side ST of the other equilateral Triangle cuts off BC the fourth Part of the Axis BK, NO will be to BO, as BK is to CK; and by changing, as NO is to BK, ſo is BO to CK. But NO is double to BK. Therefore BO is likewiſe double to CK. Therefore, becauſe of the Similitude of the Triangles, DOF, SKT, DF and ST alſo, to wit, the Diameters of the conical Baſes, will (by 4. *l.* 6.) be in a double Proportion betwixt themſelves. Wherefore, ſeeing the Cones DOF, SKT, be like, and conſequently (by 12. *l.* 12.) their Proportion is triplicate to the Proportion of the Diameters DF and ST; which is that of 2 to 1, the Cone DOF will be to the Cone SKT as 8 to 1. *Q. E. D.*

PROP. XLIV. Theorem.

Fig. 28. A *Sphere hath the ſame Proportion both in reſpect of Solidity and Surface to the equilateral Cone DOF circumſcrib'd about it, which 4 hath to 9.*

The Sphere TP is (by 42. of this) to the equilateral Cone SKT inſcribed in it, as 32 is to 9. But (by the foregoing) SKT the equilateral Cone inſcribed is to DOF the equilateral

Cone circumſcribed as 1 is to 8, that is, 9 to 72. Therefore, by Equality of Proportion, the Sphere T P is to D O F the equilateral Cone circumſcrib'd as 32 is to 72, that is, as 4 to 9. But, in *Prop.* 40, we demonſtrated that the Superficies of a Sphere is to the whole Superficies of an equilateral Cone circumſcribed, as 4 is to 9. Therefore a Sphere both in Solidity and Superficies is to an equilateral Cone circumſcrib'd about it, as 4 is to 9. *Q. E. D.*

That therefore which *Archimedes* was ſurpriz'd at, in a Sphere and Cylinder encompaſſing it, we have alſo now demonſtrated in a Sphere and an equilateral Cone encompaſſing, to wit, that there is the ſame rational Proportion of the Solidities betwixt themſelves, which there is of the Surfaces. For, as he found that the Sphere is to the Cylinder, as well in Solidity as Superficies, as 2 to 3; ſo we have now taught, that the Sphere is in reſpect both of Solidity and Surface to an equilateral Cone encompaſſing, as 4 to 9.

But from hence we ſhall, without much Labour, demonſtrate, that the very Proportion, to wit, the ſeſquialteral, which *Archimedes* ſhew'd to be betwixt the Sphere and Cylinder, is continued by the equilateral Cone circumſcrib'd both in the Solidity and Superficies; and ſo we ſhall put an End to the preſent Work.

PROP. XLV. Theorem.

AN equilateral Cone circumſcrib'd about a Sphere, and a right Cylinder in like manner circumſcrib'd about the ſame Sphere, and the ſame Sphere itſelf, continue

See the Figure prefixed to this Treatiſe.

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the ſame Proportion; to wit, the ſeſquialteral, as well in reſpect of the Solidity, as of the whole Superficies.

For, by 32. of this Book, the right Cylinder G K encompaſſing the Sphere is to the Sphere, as well in reſpect of Solidity as of the whole Superficies, as 3 is to 2, or as 6 to 4. But, by the foregoing, the equilateral Cone B A D circumſcrib'd about the Sphere is to the Sphere, in both the ſaid reſpects, as 9 is to 4. Therefore the ſame Cone is to the Cylinder, both in reſpect of Solidity and Surface, as 9 is to 6. Wherefore theſe three Bodies, a Cone, Cylinder, and Sphere, are betwixt themſelves as the Numbers 9, 6, 4, and conſequently continue the ſeſquialteral Proportion. *Q. E. D.*

[*PROP. XLVI.*

THE ſame ſeſquialteral Proportion holds betwixt an equilateral Cone and Cylinder circumſcrib'd about the ſame Sphere, in reſpect of their whole Surfaces, their ſimple Surfaces, their Solidities, Altitudes and Baſes.

This Propoſition is manifeſt, as to the whole Surfaces and Solidities, from the foregoing; as to the ſimple Surfaces, from Coroll. 3. Pr. 40. *of this; as to their Altitudes and Baſes, from* Coroll. 1 *and* 2. *of the ſame 40th Propoſition.*]

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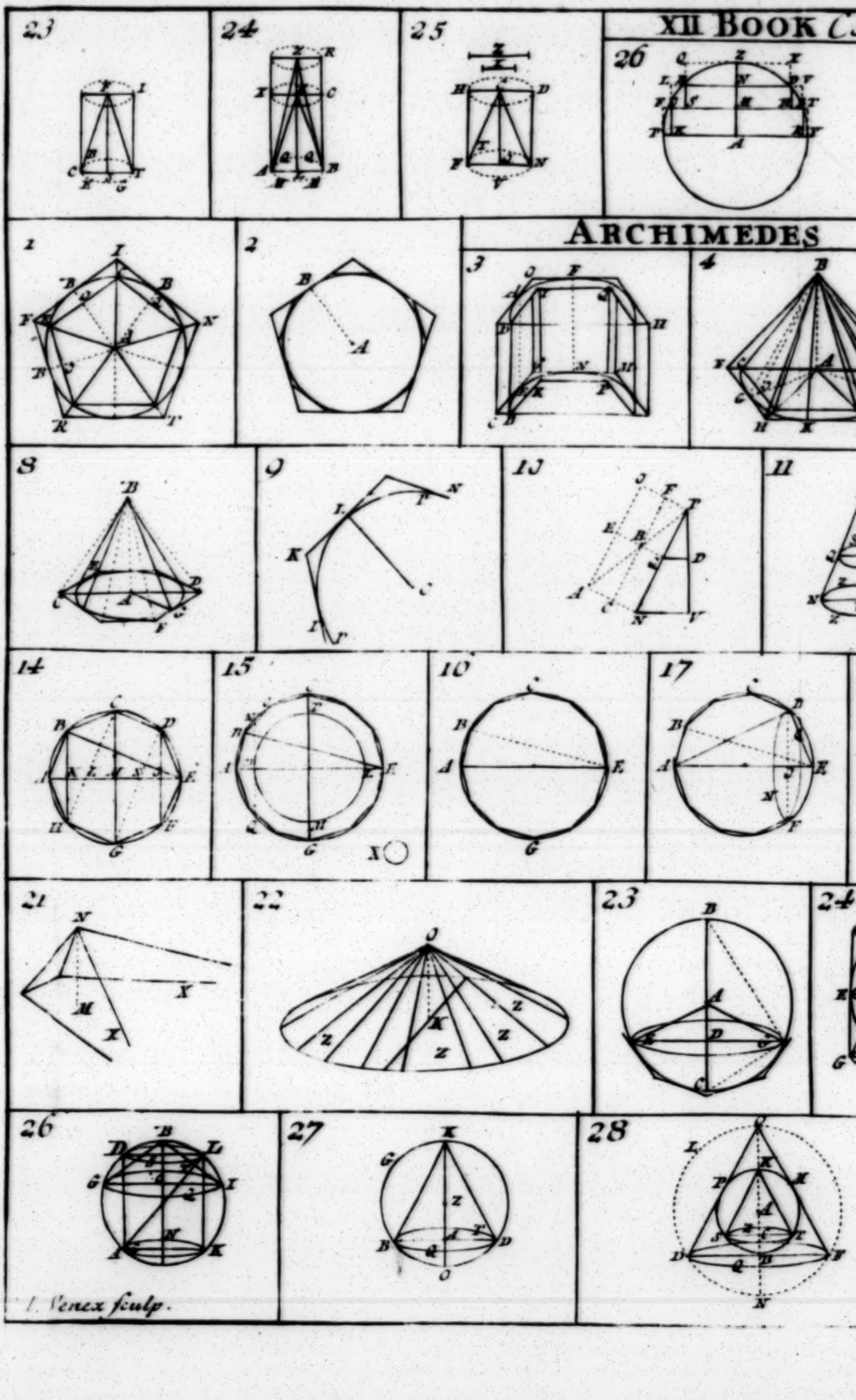

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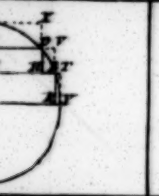

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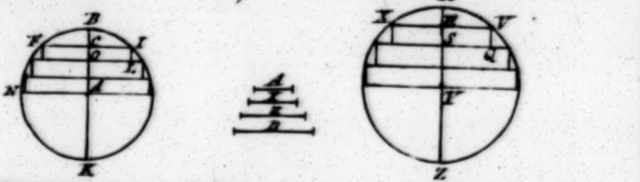

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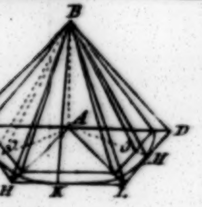

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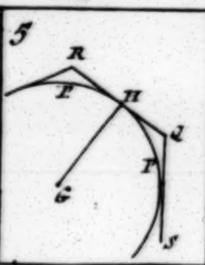

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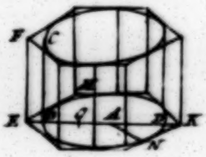

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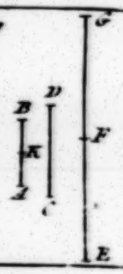

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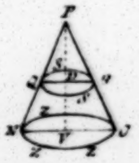

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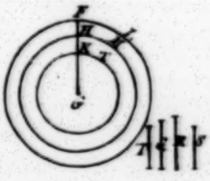

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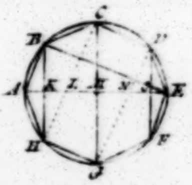

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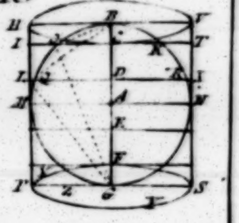

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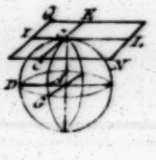

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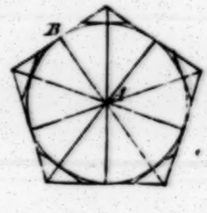

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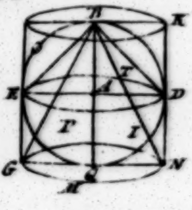

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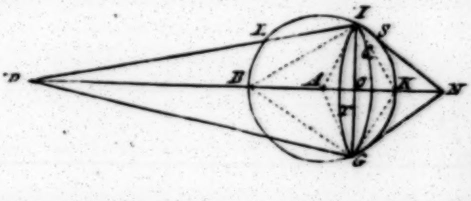

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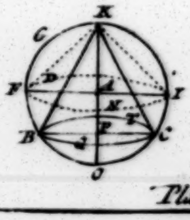

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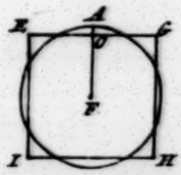

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